Real Effects of Financial Distress: The Role of Heterogeneity
Online Appendix

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A Additional Figures and Tables

A.1 Persistence of relationships

The table below shows the persistence of bank-firm relationships in Portugal. In the first two columns we report the probability of a bank being a firm's lead bank in period 't' conditional on it being the lead bank in period '$t-1'$. In columns 3 and 4 we report the probability of a particular firm borrowing from a particular bank in period 't' conditional on it having borrowed in period '$t-1'$. As we can observe, both the probabilities are in excess of 0.8 demonstrating that the relationships tend to be extremely persistent.

	$Y_t = lead_t$	$Y_t = lead_t$	$Y_t = any_t$	$Y_t = any_t$
$Y_{t-1} = lead_{t-1}$	0.802***			
	[0.000]			
$Y_{t-1} = any_{t-1}$			0.867***	
			[0.000]	
$Y_{t-1} * 2006.year$		0.827***		0.876***
		[0.000]		[0.000]
$Y_{t-1} * 2007.year$		0.810***		0.856***
		[0.000]		[0.000]
$Y_{t-1} * 2008.year$		0.818***		0.859***
		[0.000]		[0.000]
$Y_{t-1} * 2009.year$		0.760***		0.864***
		[0.000]		[0.000]
$Y_{t-1} * 2010.year$		0.795***		0.876***
		[0.000]		[0.000]
$Y_{t-1} * 2011.year$		0.792***		0.864***
		[0.000]		[0.000]
$Y_{t-1} * 2012.year$		0.810***		0.870***
		[0.000]		[0.000]
Time Effects	Y	Y	Y	Y
Number of obs.	84790059	84790059	84790059	84790059

Robust standard errors in parentheses
* p¡0.1, ** p¡0.05, *** p¡0.01

Table A1: Relationship Regression

A.2 Exploring other dimensions of heterogeneity

We have analysed firm heterogeneity along two main dimensions: leverage and maturity structure of debt. However, we also analysed differences in terms of age, size, degree of external financing, and profitability.[1] We estimate equations similar to the ones in equations (4) and (5), i.e.,

$$g^V_{i,Q4:10-Q4:09} = \alpha_0 + \alpha_1 sov_{j,Q4:09} + \alpha_2 sov_{j,Q4:09} * (high``x") + \alpha_3(high``x")$$

$$+\Gamma^1_j F_j + \Gamma^2_j B_j + \beta^{ind}_1 + \epsilon_j,$$

where $high``x"$ is a dummy and is equal to 1 for the top quartile of the respective variable, at the pre-crisis level and $x\epsilon(size, age, externalfinance, profitability)$. $high\_size = 1$ if the firm has assets of more than 1 million euros, $high\_age = 1$ if the firm is more than 18 years old, $high\_extfin = 1$ if the firm finances more than 35% of its capital expenditure through external financing, and $high\_profit = 1$ if the firm's profits as a ratio of total assets is greater than 36%. Figure 6 plots $\alpha_1 + \alpha_2$ along with the 95% confidence intervals. As can be seen, we do not find statistically significant effects for any of the variables considered.

[1] External finance = (capex-cash flows)/capex.

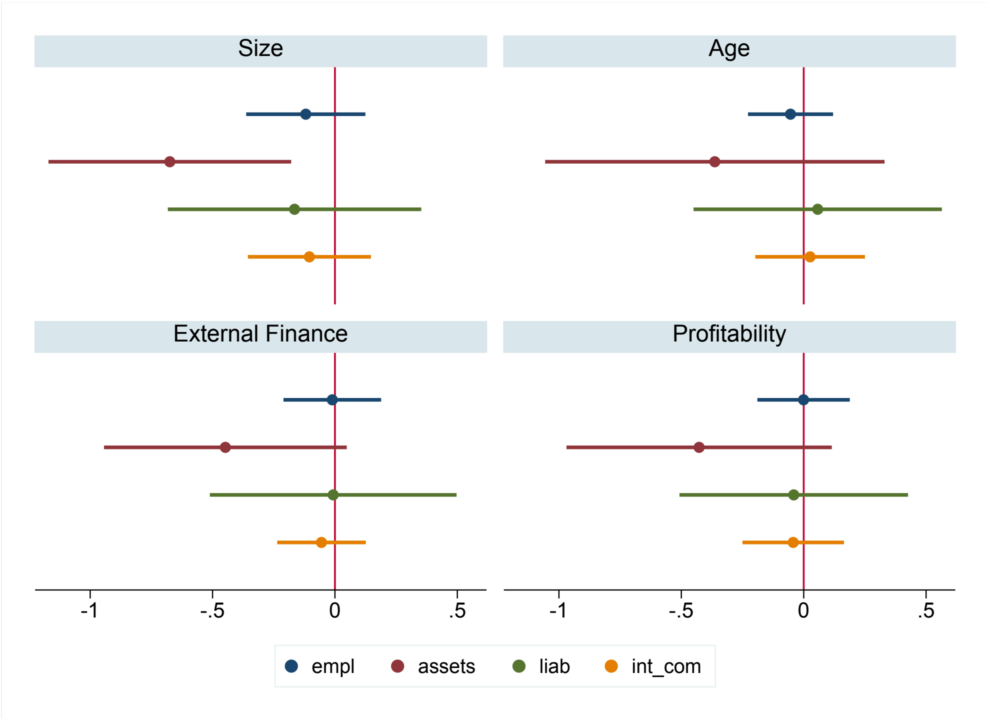


Figure 6: Exploring other dimensions of heterogeneity

B Robustness/Discussion of Results

B.1 Do the results persist over time?

The results presented in the main text correspond to the cross section of Q4:09 and Q4:10, i.e. in the immediate aftermath of the shock. A natural question to ask is if these effects existed prior to the shock or if the results continue to prevail over time i.e, after the shock. To do this, we roll out our window and estimate separate regressions in which the growth rates have been taken from 2009. Figure A1 plots the effect on the high leverage and the high short-term debt firms.

We also report backward looking regressions to demonstrate parallel trends in the same figure. In the figures, the coefficients for t+1 correspond to the interaction terms in Table 5 in the main text.[2] Each point represents α_2 or α_4 in terms of the coefficients of equation (2) in the main text.

We obtain mostly insignificant results prior to the event date (demonstrating parallel trends) and in the immediate aftermath of the shock, we document significant negative effects on firms with high leverage or high short-term debt. With respect to the results persisting over time, the broad message in these figures is that the effects on liabilities seem to have turned a corner but the effects on real variables tend to be more persistent. One of the main reasons for the observed pattern of the liabilities is the fact that Portugal entered the is the EU-ECB-IMF financial assistance program in early 2011. Central bank funding, bank capitalisations, and structural reforms all meant that credit conditions eased and had positive effects on firms' performance. It must be highlighted that we restrict our main quantitative results to the cross section before Portugal entered the bailout programme. A number of Euro level measures taken by the ECB coupled with frequent domestic regulation changes, post 2011, make identification especially difficult in this time period. It is for this reason that we present these figures mainly for illustrative purposes.

[2] We consider $t = 2009$ as the event date and report the effects leading up to the crisis and after the shock. Firms are classified as high leveraged or having high short-term debt in 2009 and we hold this fixed.

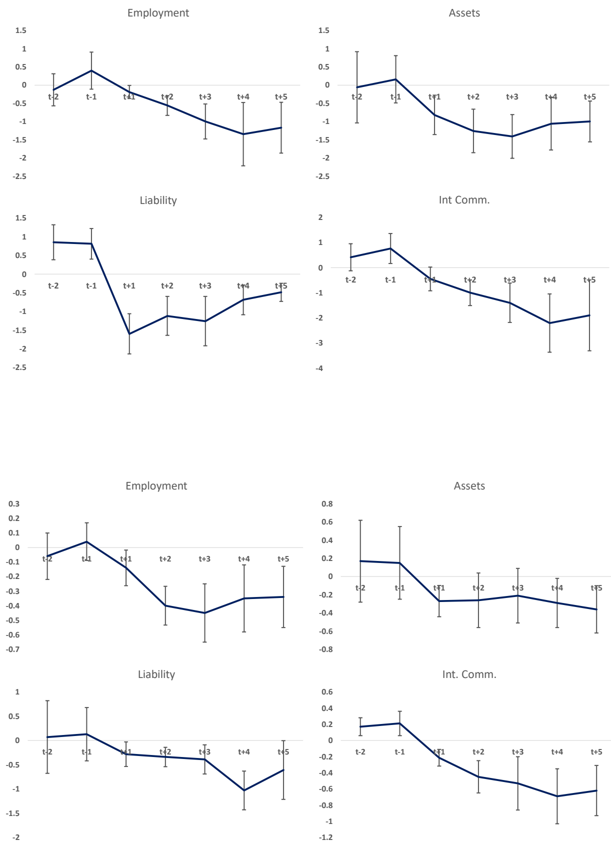


Figure A1: Sovereign Channel: Effects over time (Leverage (top) & ST Debt (bottom))

B.2 What about exposure to the sovereign debt of GIIPS?

Thus far we have considered the exposure of the banks only to the Portuguese sovereign and arguably this was the most important source of risk for the Portuguese banks. However, one can argue that a broader measure of *ex ante* vulnerability could be constructed by allowing for the exposure to the sovereign debt of the GIIPS countries.[3] To this effect, we now construct a firm level sovereign exposure variable, as before, allowing for the sovereign debt holdings for the GIIPS countries. Tables A2 and A3 highlight the fact that our previous results are robust to this alternative exposure measure. Similar checks were undertaken with the banks' holding of Portuguese and Greek debt and Portuguese and Spanish debt. In all these cases, our results and conclusions remain unaltered.

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_GIIPS (α_1)	0.010	-0.159	0.292	0.031
	(0.065)	(0.214)	(0.121)	(0.060)
Wtd_GIIPS*Highlev (α_2)	-0.179*	-0.758***	-1.447***	-0.410***
	(0.105)	(0.172)	(0.338)	(0.122)
Highlev	0.023***	-0.010	0.000	0.050
	(0.008)	(0.162)	(0.027)	(0.085)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FE	Y	Y	Y	Y
P($\alpha_1 + \alpha_2 < 0$)	0.95	0.99	0.99	0.99
Observations	88,204	89,410	89,466	89,823

Table A2: Interaction with leverage (GIIPS exposure)

Note: The dependant variables are the growth rates of employment, fixed assets, liabilities, and usage of intermediate commodities, respectively. The main independent variable is the weighted GIIPS sovereign bond holdings of firms in September 2009. Firm level controls include age, size, value added, and sector and location fixed effects. Weighted bank controls include capital ratio, liquidity ratio, and average interest rates charged by the respective banks. Clustered standard errors (bank level) are reported in parentheses. We also report the p-values from a one sided t-test with H0: $\alpha_1 + \alpha_2 < 0$. *** p<0.01, ** p<0.05, * p<0.1.

[3]Greece, Ireland, Italy, Portugal, and Spain.

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_GIIPS (α_1)	0.002	-0.244	0.155	-0.001
	(0.072)	(0.220)	(0.290)	(0.072)
Wtd_GIIPS * High_stdebt (α_2)	-0.129**	-0.242*	-0.269**	-0.204***
	(0.052)	(0.122)	(0.100)	(0.037)
High_stdebt	-0.023	-0.145	0.098***	0.000
	(0.017)	(0.160)	(0.036)	(0.044)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FE	Y	Y	Y	Y
$P(\alpha_1 + \alpha_2 < 0)$	0.99	0.97	0.99	0.99
Observations	88,204	89,410	89,828	89,823

Table A3: Interaction with ST Debt (GIIPS exposure)

Note: The dependant variables are the growth rates of employment, fixed assets, liabilities, and usage of intermediate commodities, respectively. The main independent variable is the weighted GIIPS sovereign bond holdings of firms in September 2009. Firm level controls include age, size, value added, and sector and location fixed effects. Weighted bank controls include capital ratio, liquidity ratio, and average interest rates charged by the respective banks. Clustered standard errors (bank level) are reported in parentheses. We also report the p-values from a one sided t-test with H0: $\alpha_1 + \alpha_2 < 0$. *** p<0.01, ** p<0.05, * p<0.1.

B.3 What about analysing alternative time windows?

The next robustness check was done with respect to the selection of the time window. We compute growth rates between Q4:09 and Q4:10 and this is our main window of analysis. However, we also conducted our analysis for Q4:08 and Q4:11 and also by taking growth rates of the average values of Q4:08 and Q4:09 and Q4:10 and Q4:11. Once again, our results and conclusions remain qualitatively unaltered. The results are reported in Tables A4 and A5. One of the principle reasons for not including 2011 in the baseline analysis is that 2011 was a very eventful year in terms of many influential events occurring simultaneously, e.g. Portugal requested the Eurosystem bailout, the EBA conducted the stress tests and the capital exercise, and so on.

B.4 Are the results being driven by a particularly vulnerable sector?

We also verify that our results are not driven by one particular sector. When one thinks about which sectors could be relatively more adversely affected by the sovereign debt crisis, construction seems to be the most natural candidate. Although we have sector fixed effects in of all our regressions, we re-estimated our regressions excluding the firms in the construction sector and our results hold even in that sub-sample.

B.5 Considering a broader measure of vulnerability

We also broadened our measure of risk on the banks' balance sheets by constructing a vulnerability index for the banks. This was simply the total amount of GIIPS bond holdings and the total amount of lending to the construction sector, as a fraction of total assets. Our results remain robust even to this broad vulnerability measure. To further account for the pre-existing risk on banks' balance sheets, in terms of non-performing loans, we controlled for bank-level risk by using the estimates obtained in step 2 of section 4.1 above. The results are completely robust to the inclusion of such additional controls.

B.6 How do foreign banks influence the analysis?

One could also argue that the Portuguese banking system consists of branches or subsidiaries of foreign banks which could be "bailed out" by the mother bank should

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_sov_holding	0.285	-0.572	-0.423	0.504**
	(0.173)	(0.682)	(0.613)	(0.230)
Wtd_sov_holding*Highlev	-1.447***	-1.890***	-0.895***	-1.787***
	(0.270)	(0.379)	(0.330)	(0.381)
Highlev	-0.144***	-0.048*	0.055***	0.030
	(0.031)	(0.025)	(0.014)	(0.020)
Constant	0.138***	-1.473***	0.325***	0.078***
	(0.023)	(0.067)	(0.041)	(0.023)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FEs	Y	Y	Y	Y
Observations	68,582	68,702	68,942	69,205
R-squared	0.061	0.191	0.034	0.096

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_sov_holding	0.033	-0.795	-0.054	0.204
	(0.211)	(0.705)	(1.368)	(0.312)
Wtd_sov_holding*High_stdebt	-0.446***	-1.023***	-1.540**	-0.520***
	(0.152)	(0.371)	(0.247)	(0.156)
High_stdebt	-0.054**	-0.247**	0.055*	-0.028
	(0.023)	(0.106)	(0.028)	(0.026)
Constant	0.133***	-1.492***	0.563***	0.071***
	(0.025)	(0.072)	(0.122)	(0.023)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FEs	Y	Y	Y	Y
Observations	63,878	63,963	64,428	64,428
R-squared	0.049	0.137	0.023	0.078

Table A4: Interactions with leverage and short-term debt (Q4:08 - Q4:11)

Note: This table is comparable to columns 1-8 of Table 5 in the main text. The main differences are that the weighted sovereign bond holdings of firms are kept constant at Q4:2008 and the growth rates are computed between 2008 and 2011. The dependant variables are the growth rates of employment, fixed assets, liabilities, and usage of intermediate commodities, respectively. The firm level controls used were age, size, value added, and fixed effects for the sector and location of operation. The weighted bank controls used were the capital ratio, liquidity ratio, and average loan interest rates charged. Clustered standard errors (bank level) are reported in parentheses. *** p<0.01, ** p<0.05, * p<0.1.

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_sov_holding	0.087	-0.492	0.147	0.372*
	(0.128)	(0.470)	(0.467)	(0.189)
Wtd_sov_holding*Highlev	-0.912***	-1.412***	-1.679***	-1.217***
	(0.197)	(0.252)	(0.394)	(0.252)
Highlev	-0.034*	0.274***	0.043***	0.068***
	(0.018)	(0.025)	(0.014)	(0.015)
Constant	0.107***	-0.805***	0.244***	0.048**
	(0.019)	(0.048)	(0.031)	(0.021)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FEs	Y	Y	Y	Y
Observations	68,582	68,702	68,942	69,205
R-squared	0.048	0.139	0.036	0.080

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_sov_holding	-0.067	-0.689	-0.277	0.162
	(0.157)	(0.496)	(0.474)	(0.235)
Wtd_sov_holding*High_stdebt	-0.279**	-0.516*	-0.195*	-0.298***
	(0.113)	(0.279)	(0.174)	(0.082)
High_stdebt	-0.044**	-0.162**	0.078***	-0.004
	(0.018)	(0.075)	(0.017)	(0.021)
Constant	0.103***	-0.815***	0.220***	0.041*
	(0.019)	(0.051)	(0.048)	(0.020)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FEs	Y	Y	Y	Y
Observations	63,878	63,963	64,428	64,428
R-squared	0.049	0.137	0.023	0.078

Table A5: Interactions with leverage and short-term debt (Avg (08 - 09) vs. Avg (10 - 11)

Note: This table is comparable to columns 1-8 of Table 5 in the main text. The main differences are that the weighted sovereign bond holdings of firms are kept constant at Q4:2008 and the growth rates are computed between the average values for 2008 and 2009 and 2010 and 2011. The dependant variables are the growth rates of employment, fixed assets, liabilities, and usage of intermediate commodities, respectively. The firm level controls used were age, size, value added, and fixed effects for the sector and location of operation. The weighted bank controls used were the capital ratio, liquidity ratio, and average loan interest rates charged. Clustered standard errors (bank level) are reported in parentheses. *** p<0.01, ** p<0.05, * p<0.1.

they be in distress. It must be mentioned here that the Portuguese loan market is dominated by Portuguese banks and that, as a result, the above concern is not a valid one in our analysis. Despite that, to convince the reader we address this concern by re-estimating our regression models excluding all foreign entities operating in Portugal and our results remain consistent to this specification as well. The results are reported below in Table A6.

B.7 Do banks that are more exposed to the sovereign have riskier clients?

Further analysis was conducted to ensure that our results are not driven by some particular way in which banks might be operating. For example, could it be the case that banks that were lending to riskier borrowers were also holding a high amount of “safe” sovereign debt? This could be justified as a case of diversification of the banks’ portfolio. To verify that this was not the case, we constructed bank level risk measures (share of non-performing loans in total loans), from the credit registry, and computed the correlations with sovereign holdings, *ex ante*. Figure 9 below discourages the diversification scenario. We report scatter plots and correlation coefficients in the four quarters prior to the sovereign shock. The correlations were found to be weak and non-significant. Despite this analysis, we augmented all of our regressions with sector and location specific fixed effects because such (hypothetical) matching might take place if the firm and the bank were present in a particular sector or location.

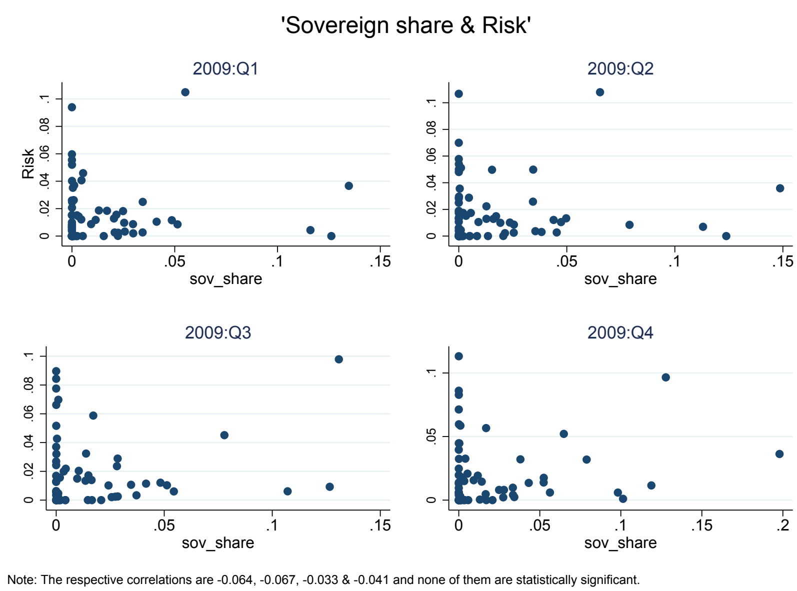


Figure 9: Bank sovereign shares vs. risk

B.8 Using an alternative estimation methodology

In terms of estimation methodology, our robustness analysis included estimating weighted least square models in which observations were weighted by some firm characteristics.

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_sov_holding	0.001	-0.251	0.390**	0.030
	(0.062)	(0.232)	(0.157)	(0.076)
Wtd_sov_holding*Highlev	-0.150**	-0.696***	-1.443***	-0.346***
	(0.074)	(0.166)	(0.405)	(0.101)
Highlev	0.017	0.033	0.010	0.048
	(0.061)	(0.139)	(0.031)	(0.091)
Constant	0.172***	-0.457***	0.106***	0.095***
	(0.016)	(0.047)	(0.024)	(0.015)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FEs	Y	Y	Y	Y
Observations	65,746	66,608	66,619	66,893
R-squared	0.034	0.087	0.037	0.056

VARIABLES	(1) Gr_emp	(2) Gr_ast	(3) Gr_liab	(4) Gr_int
Wtd_sov_holding	-0.010	-0.340	0.420	0.001
	(0.062)	(0.238)	(0.433)	(0.084)
Wtd_sov_holding*High_stdebt	-0.091	-0.166**	-0.218*	-0.190***
	(0.080)	(0.077)	(0.126)	(0.047)
High_stdebt	-0.272	-5.239***	-1.173	-1.129**
	(0.518)	(1.308)	(0.776)	(0.540)
Constant	0.170***	-0.470***	0.081	0.092***
	(0.016)	(0.048)	(0.051)	(0.016)
Firm Controls	Y	Y	Y	Y
Wtd. Bank Controls	Y	Y	Y	Y
Sector & Location FEs	Y	Y	Y	Y
Observations	65,746	66,608	66,896	66,893
R-squared	0.034	0.087	0.020	0.056

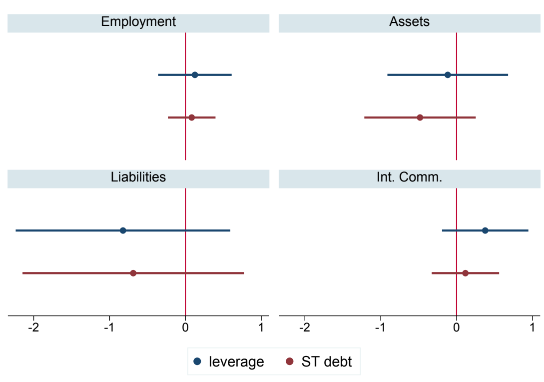
Table A6: Interactions with leverage and short-term debt (Portuguese banks only)

Note: This table is comparable to columns 1-8 of Table 5 in the main text. The only difference is that all the foreign banks have been excluded from the analysis. The dependant variables are the growth rates of employment, fixed assets, liabilities, and usage of intermediate commodities, respectively. The firm level controls used were age, size, value added, and fixed effects for the sector and location of operation. The weighted bank controls used were the capital ratio, liquidity ratio, and average loan interest rates charged. Clustered standard errors (bank level) are reported in parentheses. *** p<0.01, ** p<0.05, * p<0.1.

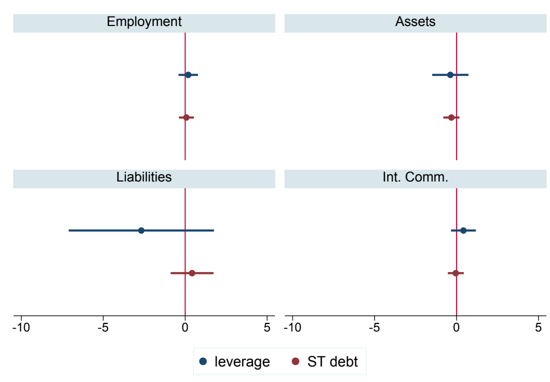
We used three different sets of weights, namely the number of employees as a measure of firm size, the total assets as an additional proxy for size, and the importance of the firm in the credit market.[4] Our results and conclusions remain completely robust to these weighted specifications as well.

B.9 Placebo regressions

We also carry out some placebo exercises to convince the reader that the effects documented are indeed a feature of this particular stress period and are not confounded by other factors. In the regressions documented thus far, we hold the bank's sovereign exposures constant at their 2009:Q4 levels and report growth rates between 2009:Q4 and 2010:Q4. To be precise, we carry out two placebo exercises: (i) hold the sovereign shares constant in 2007:Q4 and analyse growth rates between 2007:Q4 and 2008:Q4 and (ii) hold the sovereign shares constant at 2008:Q4 and analyse growth rates between 2008:Q4 and 2009:Q4. In other words, we recreate columns 1-8 of Table 5 but calculating the growth rates between 2007 and 2008 (Figure 10 panel (a)) and between 2008-2009 (Figure 10 panel (b)). We do not find any significant effects for the highly leveraged firms or the firms that had a greater share of short-term debt for any of the firm outcome variables under consideration. This lends further credence to the fact that the results presented are specific to the period under consideration.



(a) $\%\Delta(2007-2008)$



(b) $\%\Delta(2008-2009)$

Figure 10: Placebo regressions

[4]For the last case, the weight a firm received was its share of borrowing as a fraction of total borrowing by all firms in the sample.

C General CRRA Case

This appendix extends the analysis in the main text to the case of a general CRRA utility function. In this case, the optimal investment decision solves

$$\max_k \mathbb{E}_z \left\{ \frac{c_2^{1-\sigma}}{1-\sigma} \right\}$$

where

$$c_2 = \left(z - 1 - r_1^1\right) k + y_2 + \left(1 + r_1^1\right) y_1 - \left(1 + r_1^1\right) \left(1 + r_0^1\right) \left(d_0 - d_0^2\right) - \left(1 + r_0^2\right) d_0^2.$$

The first order condition of this problem is given by

$$\mathbb{E} \left\{ c_2^{-\sigma} \left(z - 1 - r_1^1\right) \right\} = 0$$

or

$$E \left\{ \left[\left(z - 1 - r_1^1\right) k + y_2 + \left(1 + r_1^1\right) y_1 - \left(1 + r_1^1\right) \left(1 + r_0^1\right) \left(d_0 - d_0^2\right) - \left(1 + r_0^2\right) d_0^2 \right]^{-\sigma} \left(z - 1 - r_1^1\right) \right\} = 0.$$

Guessing a linear solution of the form

$$k = \bar{k}\left(r_1^1\right) \underbrace{\left[\frac{y_2}{1 + r_1^1} + y_1 - \left(1 + r_0^1\right) \left(d_0 - d_0^2\right) - \frac{1 + r_0^2}{1 + r_1^1} d_0^2 \right]}_{\omega\left(r_1^1, d_0^2\right)},$$

we obtain

$$\mathbb{E}_z \left\{ \left[\left(z - 1 - r_1^1\right) \bar{k}\left(r_1^1\right) \omega + \left(1 + r_1^1\right) \omega \right]^{-\sigma} \left(z - 1 - r_1^1\right) \right\} = 0,$$

which is verified provided

$$\mathbb{E}_z \left\{ \frac{\left(z - 1 - r_1^1\right)}{\left[\left(z - 1 - r_1^1\right) \bar{k}\left(r_1^1\right) + \left(1 + r_1^1\right) \right]^{\sigma}} \right\} = 0$$

or

$$\mathbb{E}_z \left\{ \frac{\left(z-1-r_1^1\right)}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma}} \right\} = 0.$$

The impact of the cost of credit on $\bar{k}\left(r_1^1\right)$ is given by

$$\begin{aligned}
\frac{d\bar{k}\left(r_1^1\right)}{dr_1^1} &= -\frac{\mathbb{E}_z\left\{-\frac{1}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma}} - \sigma\frac{\left(z-1-r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma+1}}\right\}}{-\sigma\mathbb{E}_z\left\{\frac{\left(z-1-r_1^1\right)^2}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma}}\right\}} \\
&= -\frac{\mathbb{E}_z\left\{\frac{1}{\sigma}\frac{1}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma}} + \frac{\left(z-1-r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma+1}}\right\}}{\mathbb{E}_z\left\{\frac{\left(z-1-r_1^1\right)^2}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma}}\right\}} \\
&= -\frac{1}{\mathbb{E}_z\left\{\frac{\left(z-1-r_1^1\right)^2}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma}}\right\}} \\
&\quad \left[\frac{1}{\sigma}\mathbb{E}_z\left\{\frac{1}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma}}\right\}\right. \\
&\quad \left. +\left[1-\bar{k}\left(r_1^1\right)\right]\underbrace{\mathbb{E}_z\left\{\frac{\left(z-1-r_1^1\right)}{\left[z\bar{k}\left(r_1^1\right)+\left(1+r_1^1\right)\left(1-\bar{k}\left(r_1^1\right)\right)\right]^{\sigma+1}}\right\}}_{<0}\right].
\end{aligned}$$

When $\bar{k}\left(r_1^1\right) < 1$, there is a positive income effect associated with the increase in the interest rate. With CRRA preferences, the higher income results in a lower absolute risk aversion and, through this effect, to a higher fraction of the portfolio invested in the risky asset. The net effect depend on the strenght of the substitution and income effects. When σ is high and $\bar{k}\left(r_1^1\right) < 1$, then $d\bar{k}\left(r_1^1\right)/dr_1^1 > 0$. For the log case in the main text, the substitution effect dominates. If $\bar{k}\left(r_1^1\right) > 1$, the income effect associated with an increase in the interest rate is negative. Thus, in this case $d\bar{k}\left(r_1^1\right)/dr_1^1 < 0$.[5]

[5] For the case of a general utility function $u(.)$, the optimal decision to allocate a unit of investment into a risky asset with gross return z and riskless asset with gross return $1+r$ is the solution to the following first order condition

$$E_z\left[(z-1-r)\,u'\left(\alpha z+(1-\alpha)(1+r)\right)\right] = 0.$$

Consumption in the last period can be written as

$$
\begin{aligned}
c_2\left(z, r_1^1, d_0^2\right) &= \left[\left(\frac{z-1-r_1^1}{1+r_1^1}\right)\bar{k}\left(r_1^1\right)+1\right] \\
&\quad \left[y_2+\left(1+r_1^1\right)y_1-\left(1+r_1^1\right)\left(1+r_0^1\right)\left(d_0-d_0^2\right)-\left(1+r_0^2\right)d_0^2\right] \\
&= \bar{c}\left(z, r_1^1\right)\underbrace{\left[y_2+\left(1+r_1^1\right)y_1-\left(1+r_1^1\right)\left(1+r_0^1\right)\left(d_0-d_0^2\right)-\left(1+r_0^2\right)d_0^2\right]}_{\left(1+r_1^1\right)\omega\left(r_1^1, d_0^2\right)}
\end{aligned}
$$

where

$$
\frac{\partial \bar{c}\left(z, r_1^1\right)}{\partial r_1^1} < 0
$$

provided $d\bar{k}\left(r_1^1\right)/dr_1^1 < 0$.

The optimal maturity choice solves

$$
\max_{d_0^2} \frac{1}{1-\sigma}\mathbb{E}_{r_1^1}\left\{\mathbb{E}_z\left[c_2\left(z, r_1^1, d_0^2\right)^{1-\sigma}\right]\right\}
$$

The first order condition is given by

$$
\mathbb{E}_{r_1^1}\left\{\mathbb{E}_z\left[c_2\left(z, r_1^1 d_0^2\right)^{-\sigma}\frac{\partial c_2\left(z, r_1^1, d_0^2\right)}{\partial d_0^2}\right]\right\} = 0
$$

or

$$
\begin{aligned}
\mathbb{E}_{r_1^1}\left\{\left[\frac{\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^\sigma}\right]\right. \\
\left.\mathbb{E}_z\left[\bar{c}\left(z, r_1^1\right)^{1-\sigma}\right]\right\} &= 0.
\end{aligned}
$$

To characterise the choice of long-term debt it is important to understand how the interest rate in the interim period affects the marginal value of long-term debt. The interest rate in the interim period affects the marginal value of holding long-term debt

---

In this case,

$$
\frac{d\alpha}{dr} = -\frac{-E_z\left\{u'\left(\alpha z+(1-\alpha)(1+r)\right)\left[1-(1-\alpha)(z-1-r)\frac{u''(\alpha z+(1-\alpha)(1+r))}{u'(\alpha z+(1-\alpha)(1+r))}\right]\right\}}{E_z\left[(z-1-r)^2 u''\left(\alpha z+(1-\alpha)(1+r)\right)\right]}.
$$

For the case of a CARA utility function, the income effect associated with an increase in the riskless rate vanishes. Provided $\alpha < 1$, the income effect is positive when the coefficient of absolute risk aversion is decreasing, as in the CRRA case.

through four channels: (i) it affects the marginal utility of consumption by impacting the net worth in the interim period, $\left[\left(1+r_1^1\right)\omega\left(r_1^1, d_0^2\right)\right]^{-\sigma}$; (ii) it affects the marginal utility of consumption by impacting the consumption share out of the net-worth in the final period, $\bar{c}\left(z, r_1^1\right)^{-\sigma}$; (iii) it affects the sensitivity of consumption to a change in the amount of long-term debt through its impact on the consumption share, $\bar{c}\left(r_1^1, d_0^2\right)$; (iv) it affects the sensitivity of consumption to a change in the amount of long-term debt through its impact on the return of long term debt, $\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)$.

In the log case, the second and third channels cancel out. In this case, provided that the expectation hypothesis holds, it is optimal to set the maturity structure to perfectly hedge the cash-flow risk in the interim period stemming from the interest rate shock, $d_0^2 = d_0 - y_1/\left(1+r_0^1\right)$. If $\sigma > 1 (< 1)$, then it is optimal to be more (less) than perfectly hedged, i.e., $d_0^2 > (<) d_0 - y_1/\left(1+r_0^1\right)$. Intuitively, the demand for hedging increases when the risk aversion is higher than 1. The converse is true when the risk aversion is smaller than 1.

We next extend the analysis behind Propositions 3 for the general CRRA case. The generalisation of Proposition 4 closely follows the analysis of the log case in the main text.

The effect of the cash-flows of the long-term project on the maturity choices are

$$\frac{\partial d_0^2}{\partial y_1} = -\frac{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\left(1+r_1^1\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}} < 0$$

and

$$\frac{\partial d_0^2}{\partial y_2} = -\frac{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}} < 0.$$

Similarly, the effect of the cost of long-term debt on the maturity choice is given by

$$
\begin{aligned}
\frac{\partial d_0^2}{\partial\left(1+r_0^2\right)} &= -\frac{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}} \\
&\quad +(\sigma-1)d_0^2\frac{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,r_1^1\right)^{1-\sigma}\right]\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}.
\end{aligned}
$$

This derivative is negative if assumption 2 is satisfied and σ is not too large. As discussed earlier, the effect a change in the return has a substitution and a countervailing income effect. The income effect is stronger the larger the value of σ.

We first consider the case in which entrepreneurs are heterogeneous with respect to the initial leverage d_0 and the income in the interim period y_1. The differential effect of the interest rate shock when the maturity structure changes is given by

$$
\begin{aligned}
\frac{\partial^2 \log \hat{k}}{\partial r_1^1 \partial d_0^2} &= \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_1} \frac{\partial y_1}{\partial d_0^2} \\
&= \frac{(1+r_0^1)}{\omega^2} \frac{y_2 - (1+r_0^2)\left(d_0 - \frac{y_1}{1+r_0^1}\right)}{(1+r_1^1)^2} \\
&\quad - \frac{1}{\omega^2} \frac{y_2 - (1+r_0^2) d_0^2}{(1+r_1^1)^2} \frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\left(1+\tilde{r}_1^1\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}. \\
&= \frac{1}{\omega^2} \frac{1}{(1+r_1^1)^2} \frac{1}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\left(1+\tilde{r}_1^1\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left((1+r)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}} \\
&\quad \mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right. \\
&\quad \left[\left(1+r_0^1\right)\left(y_2-\left(1+r_0^2\right)\left(d_0-\frac{y_1}{1+r_0^1}\right)\right)\left(1+\tilde{r}_1^1\right)\right. \\
&\quad \left.\left. - \left(y_2-(1+r_0^2)d_0^2\right)\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\right]\right\} \\
&= \frac{1}{\omega^2} \frac{1+r_0^2}{(1+r_1^1)^2} \frac{1}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\left(1+\tilde{r}_1^1\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}} \\
&\quad \mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma}}\right\} \\
&= 0.
\end{aligned}
$$

The differential effect of the interest rate shock when leverage changes is given by

$$
\begin{aligned}
\frac{\partial^2 \log \hat{k}}{\partial r_1^1 \partial d_0} &= \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_1} \frac{\partial y_1}{\partial d_0} \\
&= -\frac{(1+r_0^1)}{\omega^2} \frac{y_2 - (1+r_0^2)(d_0^2)}{(1+r_1^1)^2} + \frac{1}{\omega^2} \frac{y_2 - (1+r_0^2) d_0^2}{(1+r_1^1)^2} (1+r_0^1) \\
&= 0.
\end{aligned}
$$

We next consider the case in which entrepreneurs are heterogeneous with respect to the initial leverage d_0 and the income in the interim period y_2.

The differential effect of the interest rate shock when the maturity structure changes is given by

$$
\begin{aligned}
\frac{\partial^2 \log \hat{k}}{\partial r_1^1 \partial d_0^2} &= \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_2} \frac{\partial y_2}{\partial d_0^2} \\
&= \frac{(1+r_0^1)}{\omega^2} \frac{y_2 - (1+r_0^2)(d_0 - \frac{y_1}{1+r_0^1})}{(1+r_1^1)^2} \\
&\quad - \left[\frac{1}{\omega^2} \frac{y_2 - (1+r_0^2) d_0^2}{(1+r_1^1)^2} \frac{1}{1+r_1^1} - \frac{1}{\omega} \frac{1}{(1+r_1^1)^2} \right] \\
&\quad \frac{\mathbb{E}_{\tilde{r}_1^1} \left\{ \frac{\mathbb{E}_z \left[\bar{c}\left(z, \tilde{r}_1^1\right)^{1-\sigma} \right] \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right)^2}{\left[y_2 + \left(1+\tilde{r}_1^1\right)\left(y_1 - \left(1+r_0^1\right) d_0\right) + \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right) d_0^2 \right]^{\sigma+1}} \right\}}{\mathbb{E}_{\tilde{r}_1^1} \left\{ \frac{\mathbb{E}_z \left[\bar{c}\left(z, \tilde{r}_1^1\right)^{1-\sigma} \right] \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right)}{\left[y_2 + \left(1+\tilde{r}_1^1\right)\left(y_1 - \left(1+r_0^1\right) d_0\right) + \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right) d_0^2 \right]^{\sigma+1}} \right\}}.
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1+r_0^1)}{\omega^2} \frac{y_2 - (1+r_0^2)(d_0 - \frac{y_1}{1+r_0^1})}{(1+r_1^1)^2} \\
&\quad - \frac{1}{\omega^2} \frac{1}{(1+r_1^1)^2} \left[-y_1 + \left(1+r_0^1\right) d_0 - \left(1+r_0^1\right) d_0^2 \right] \\
&\quad \frac{\mathbb{E}_{\tilde{r}_1^1} \left\{ \frac{\mathbb{E}_z \left[\bar{c}\left(z, \tilde{r}_1^1\right)^{1-\sigma} \right] \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right)^2}{\left[y_2 + \left(1+\tilde{r}_1^1\right)\left(y_1 - \left(1+r_0^1\right) d_0\right) + \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right) d_0^2 \right]^{\sigma+1}} \right\}}{\mathbb{E}_{\tilde{r}_1^1} \left\{ \frac{\mathbb{E}_z \left[\bar{c}\left(z, \tilde{r}_1^1\right)^{1-\sigma} \right] \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right)}{\left[y_2 + \left(1+\tilde{r}_1^1\right)\left(y_1 - \left(1+r_0^1\right) d_0\right) + \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right) d_0^2 \right]^{\sigma+1}} \right\}}
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
&= \frac{1}{\omega^2} \frac{1+r_0^1}{(1+r_1^1)^2} \frac{1}{\mathbb{E}_{\tilde{r}_1^1} \left\{ \frac{\mathbb{E}_z \left[\bar{c}\left(z, \tilde{r}_1^1\right)^{1-\sigma} \right] \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right)}{\left[y_2 + \left(1+\tilde{r}_1^1\right)\left(y_1 - \left(1+r_0^1\right) d_0\right) + \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right) d_0^2 \right]^{\sigma+1}} \right\}} \\
&\quad \mathbb{E}_{\tilde{r}_1^1} \left\{ \frac{\mathbb{E}_z \left[\bar{c}\left(z, \tilde{r}_1^1\right)^{1-\sigma} \right] \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right)}{\left[y_2 + \left(1+\tilde{r}_1^1\right)\left(y_1 - \left(1+r_0^1\right) d_0\right) + \left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) \right) d_0^2 \right]^{\sigma}} \right\} \\
&= 0.
\end{aligned}
$$

The differential effect of the interest rate shock when leverage changes is given by

$$
\begin{aligned}
\frac{\partial^2 \log \hat{k}}{\partial r_1^1 \partial d_0} &= \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_2} \frac{\partial y_2}{\partial d_0} \\
&= -\frac{(1+r_0^1)}{\omega^2} \frac{y_2 - (1+r_0^2)(d_0^2)}{(1+r_1^1)^2} \\
&\quad - \frac{1}{\omega^2} \frac{1}{(1+r_1^1)^2} \left[-y_1 + \left(1+r_0^1\right) d_0 - \left(1+r_0^1\right) d_0^2\right] \\
&\quad \left(1+r_0^1\right) \frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\left(1+\tilde{r}_1^1\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}}.
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
&= \frac{1}{\omega^2} \frac{1+r_0^1}{(1+r_1^1)^2} \frac{1}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma+1}}\right\}} \\
&\quad \mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\mathbb{E}_z\left[\bar{c}\left(z,\tilde{r}_1^1\right)^{1-\sigma}\right]\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^{\sigma}}\right\} \\
&= 0.
\end{aligned}
$$

D Richer Aggregate Shock, $y_1\left(r_1^1\right)$

In this appendix we characterise a generalisation of the model presented in the main text to allow for the cash flow in period 1 to be a decreasing function of the realisation of the interest rate r_1^1, $y_1 = \bar{y}_1 + y\left(r_1^1\right)$, $\partial y\left(r_1^1\right)/\partial r_1^1 < 0$.

Given the total leverage d_0 and the quantity of long-term debt d_0^2, the investment decision in the interim period solves

$$\max_k \mathbb{E}_z \left\{\log c_2\right\}$$

where

$$\begin{aligned} c_2 \;=\;& \left(z - 1 - r_1^1\right) k + y_2 + \left(1 + r_1^1\right)\left(\bar{y}_1 + y_1\left(r_1^1\right)\right) \\ & - \left(1 + r_1^1\right)\left(1 + r_0^1\right)\left(d_0 - d_0^2\right) - \left(1 + r_0^2\right) d_0^2. \end{aligned}$$

The first-order condition is:

$$\mathbb{E}_z \left\{\frac{z - 1 - r_1^1}{c_2}\right\} = 0.$$

The solution is given by

$$\begin{aligned} & k\left(r_1^1, d_0, d_0^2, y_1, y_2, r_0^1, r_0^2\right) \\ & = \bar{k}\left(r_1^1\right)\left[\bar{y}_1 + y_1\left(r_1^1\right) - \left(1 + r_0^1\right) d_0 + \frac{y_2}{1 + r_1^1} + \left(1 + r_0^1 - \frac{1 + r_0^2}{1 + r_1^1}\right) d_0^2\right] \\ & = \bar{k}\left(r_1^1\right)\omega, \end{aligned}$$

where $\bar{k}\left(r_1^1\right)$ solves

$$\mathbb{E}_z \left[\frac{1}{\bar{k}\left(r_1^1\right) + \frac{1}{\frac{z}{1 + r_1^1} - 1}}\right] = 0,$$

with

$$\frac{\partial \bar{k}\left(r_1^1\right)}{\partial r_1^1} = -\frac{\mathbb{E}_z\left[\frac{\frac{1}{\left(\frac{z}{1+r_1^1}-1\right)^2}\frac{z}{\left(1+r_1^1\right)^2}}{\left(\bar{k}\left(r_1^1\right)+\frac{1}{\frac{z}{1+r_1^1}-1}\right)^2}\right]}{\mathbb{E}_z\left[\frac{1}{\left(\bar{k}\left(r_1^1\right)+\frac{1}{\frac{z}{1+r_1^1}-1}\right)^2}\right]} < 0,$$

and ω is the value of the net worth of the entrepreneur at the beginning of the intermediate period.

The semi-elasticity of investment with respect to the interest rate in the interim period is

$$\begin{aligned}\frac{\partial \log k\left(r_1^1\right)}{\partial r_1^1} &= \frac{1}{\bar{k}(r_1^1)}\frac{\partial \bar{k}\left(r_1^1\right)}{\partial r_1^1} - \frac{1}{\omega}\frac{\frac{y_2-(1+r_0^2)d_0^2}{1+r_1^1}}{\left(1+r_1^1\right)} \\ &+ \frac{1}{\omega}\frac{\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\left(1+r_1^1\right)}. \end{aligned} \tag{1}$$

The proofs of the counterparts of propositions 1 and 2 follow from differentiating this expression with respect to leverage and the maturity of the debt in the first period, with the caveat that in the generalised model the necessary and sufficient condition for Proposition 1 to hold is now

$$y_2 - (1+r_0^2)d_0^2 - \frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1} > 0,$$

and the result in Proposition 2 goes through with the additional (sufficient) condition

$$\left(1+r_1^1\right)\left(1+r_0^1\right) > 1 + r_0^2.$$

Differentiating (1) with respect to leverage we obtain a counterpart of the result in Proposition 1:

$$\frac{\partial^2 \log k\left(r_1^1\right)}{\partial r_1^1 \partial d_0} = -\frac{1+r_0^1}{1+r_1^1}\frac{1}{\omega}\frac{\frac{y_2-(1+r_0^2)d_0^2}{1+r_1^1} - \frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\omega(r_1^1)} < 0.$$

The inequality follows from the condition $y_2 - (1+r_0^2)d_0^2 - \frac{\partial y_1(r_1^1)}{\partial r_1^1} > 0$. This condition is weaker than the one needed in the proof of Proposition 1. Similarly, differentiating (1) with respect to the maturity of the debt we obtain a counterpart of the result in Proposition 2:

$$
\begin{aligned}
&\frac{\partial^2 \log k\left(r_1^1\right)}{\partial r_1^1 \partial d_0^2} \\
=&\frac{1}{\left[\omega(r_1^1)\right]^2\left(1+r_1^1\right)} \\
&\left[\left(1+r_0^1-\frac{1+r_0^2}{1+r_1^1}\right)\left(\frac{y_2-(1+r_0^2)d_0^2}{1+r_1^1}-\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\right)\right. \\
&\left.+\frac{1+r_0^2}{1+r_1^1}\left(y_1-\left(1+r_0^1\right)d_0+\frac{y_2}{1+r_1^1}+\left(1+r_0^1-\frac{1+r_0^2}{1+r_1^1}\right)d_0^2\right)\right] \\
=&\frac{\left(1+r_0^1\right)\left(1+r_0^2\right)}{\omega^2\left(1+r_1^1\right)^2} \\
&\left[\frac{y_2}{1+r_0^2}+\frac{y_1}{\left(1+r_0^1\right)}-d_0-\left(\frac{1+r_1^1}{1+r_0^2}-\frac{1}{\left(1+r_0^1\right)}\right)\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\right] \\
=&\frac{\left(1+r_0^1\right)\left(1+r_0^2\right)}{\omega^2\left(1+r_1^1\right)^2}\left[\frac{y_2}{1+r_0^2}+\frac{y_1}{1+r_0^1}-d_0-\left(\frac{1+r_1^1}{1+r_0^2}-\frac{1}{\left(1+r_0^1\right)}\right)\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\right]>0.
\end{aligned}
$$

where the inequality follows from Assumption 2, the assumption that $\partial y\left(r_1^1\right)/\partial r_1^1 < 0$, and the added requirement that $\left(1+r_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right) > 0$, i.e., that we are considering a relatively high realization of the interest rate shock.

For the analysis in the following section we use the expressions for the following additional cross-partials:

$$
\frac{\partial^2 \log k\left(r_1^1\right)}{\partial r_1^1 \partial \bar{y}_1} = \frac{1}{\omega^2}\frac{\frac{y_2-(1+r_0^2)d_0^2}{1+r_1^1}-\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\left(1+r_1^1\right)},
$$

$$
\begin{aligned}
\frac{\partial^2 \log k\left(r_1^1\right)}{\partial r_1^1 \partial y_2} &= -\frac{1}{\omega}\frac{1}{\left(1+r_1^1\right)^2} \\
&\quad+\frac{1}{\omega^2}\frac{\frac{y_2-(1+r_0^2)d_0^2}{1+r_1^1}-\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\left(1+r_1^1\right)^2} \\
&= \frac{1}{\omega^2}\frac{-\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}-\left(\bar{y}_1+y_1\left(r_1^1\right)-\left(1+r_0^1\right)d_0+\left(1+r_0^1\right)d_0^2\right)}{\left(1+r_1^1\right)^2},
\end{aligned}
$$

and

$$\frac{\partial^2 \log k\left(r_1^1\right)}{\partial r_1^1 \partial r_0^2} \quad = \quad \frac{1}{\omega}\frac{1}{\left(1+r_1^1\right)^2}d_0^2 - \frac{1}{\omega^2}\frac{\frac{y_2-(1+r_0^2)d_0^2}{1+r_1^1} - \frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\left(1+r_1^1\right)}d_0^2.$$

D.1 Maturity decision

Using the optimal investment decision, consumption in the last period can be written as

$$\begin{aligned} c_2 &= (z-1-r_1^1)k\left(r_1^1, d_0, d_0^2, \bar{y}_1 + y_1\left(r_1^1\right), y_2, r_0^1, r_0^2\right) \\ &\quad + y_2 - (1+r_0^2)d_0^2 + (1+r_1^1)(\bar{y}_1 + y_1\left(r_1^1\right) - (1+r_0^1)(d_0 - d_0^2)) \\ &\quad = \left[(z-1-r_1^1)\bar{k}(r_1^1) + 1 + r_1^1\right] \\ &\quad \left[\bar{y}_1 + y_1\left(r_1^1\right) - \left(1+r_0^1\right)d_0 + \frac{y_2}{1+r_1^1} + \left(1 + r_0^1 - \frac{1+r_0^2}{1+r_1^1}\right)d_0^2\right]. \end{aligned}$$

Given the investment decision in the interim period, the optimal debt maturity solves

$$\max_{d_0^2} \mathbb{E}_{r_1^1} \log\left[y_2 + \left(1+r_1^1\right)\left(\bar{y}_1 + y_1\left(r_1^1\right) - \left(1+r_0^1\right)d_0\right) + \left(\left(1+r_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right)\right)d_0^2\right]$$

with first-order condition

$$\mathbb{E}_{r_1^1}\left\{\frac{\left(1+r_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right)}{y_2 + \left(1+r_1^1\right)\left(\bar{y}_1 + y_1\left(r_1^1\right) - \left(1+r_0^1\right)d_0\right) + \left(\left(1+r_1^1\right)\left(1+r_0^1\right) - \left(1+r_0^2\right)\right)d_0^2}\right\} = 0. \tag{2}$$

Assumption 2 implies that the amount of long-term debt is a decreasing function of the term premium

$$
\begin{aligned}
&\frac{\partial d_0^2}{\partial\left(1+r_0^2\right)} \\
&= -\frac{\mathbb{E}_{r_1^1}\left\{\frac{-\left[y_2+\left(1+r_1^1\right)\left(\bar{y}_1+y_1\left(r_1^1\right)-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2}{\left[y_2+\left(1+r_1^1\right)\left(\bar{y}_1+y_1\left(r_1^1\right)-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{-\left[\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right]^2}{\left[y_2+\left(1+r_1^1\right)\left(\bar{y}_1+y_1\left(r_1^1\right)-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} \\
&= -\frac{\mathbb{E}_{r_1^1}\left\{\frac{\left[y_2+\left(1+r_1^1\right)\left(\bar{y}_1+y_1\left(r_1^1\right)-\left(1+r_0^1\right)d_0\right)\right]}{\left[y_2+\left(1+r_1^1\right)\left(\bar{y}_1+y_1\left(r_1^1\right)-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\left[\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right]^2}{\left[y_2+\left(1+r_1^1\right)\left(\bar{y}_1+y_1\left(r_1^1\right)-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} < 0. \qquad (3)
\end{aligned}
$$

Next, we consider the comparative statics of long-term debt when there is a strictly positive term premium $1+r_0^2 > \left(1+r_0^1\right)\mathbb{E}\left(1+r_1^1\right)$.

As before, the amount of long-term debt is a decreasing function of the (fixed component of the) cash flow in the interim period

$$
\begin{aligned}
\frac{\partial d_0^2}{\partial \bar{y}_1} &= -\frac{\mathbb{E}_{r_1^1}\left\{\frac{\left(1+r_1^1\right)\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2\right]^2}\right\}} \\
&= -\frac{1}{1+r_0^1}\frac{\mathbb{E}_{r_1^1}\left\{\frac{\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2+\left(1+r_0^2\right)\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} \\
&= -\frac{1}{1+r_0^1} \\
&\quad -\frac{1+r_0^2}{1+r_0^1}\frac{\mathbb{E}_{r_1^1}\left\{\frac{\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} \\
&< 0.
\end{aligned}
$$

The first term equals the effects of $\bar{y}_1$ on d_0^2 when the entrepreneur is not exposed to interest rate risk. As the cash flow in the interim period increases, more of the initial leverage can be repaid in one period and, therefore, less long-term debt needs to be issued. The sign of the second terms follows from (2) and the fact that when

$d_0^2 < d_0 - y_1/\left(1+r_0^1\right)$, i.e., short-term debt is issued in the interim period, and the fact that net worth in the interim period, $y_2 + \left(1+r_1^1\right)\left(y_1 - \left(1+r_0^1\right)\left(d_0 - d_0^2\right)\right) - \left(1+r_0^2\right)d_0^2$, is a decreasing function of r_1^1. The second term captures the effect of changes in the net worth on the demand for insurance. In general, the sign of this term depends on the coefficient of risk aversion. In our log case, the coefficient of absolute risk aversion is a strictly decreasing function of net worth. Therefore, the second term is negative.

Related, the amount of long-term debt is a decreasing function of the cash flow in the last period y_2

$$\frac{\partial d_0^2}{\partial y_2} = -\frac{\mathbb{E}_{r_1^1}\left\{\frac{\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} < 0.$$

As was the case when considering the effect of the cash flow in the interim period, as the coefficient of risk aversion is decreasing, the demand for insurance is a decreasing function of the cash flow in the last period.

We are now ready to analyse a counterpart of the result in Proposition 3. First, we consider the case in which entrepreneurs are heterogeneous with respect to the initial leverage and the component of the income in the interim period that is invariant to the interest rate shock $\bar{y}_1$. Equation (2) defines implicitly a function relating y_1 and d_0 and d_0^2, which, abusing notation, we denote $y_1(d_0, d_0^2)$. Using this notation, we can define the reduced form relationship between investment, the interest rate shock, leverage and debt maturity as

$$\hat{k}(r_1^1, d_0, d_0^2) = k(r_1^1, d_0, d_0^2, y_1(d_0, d_0^2)), \tag{4}$$

where we have omitted the dependence of k on parameters that are assumed to be common across entrepreneurs, i.e., y_2, r_0^1, r_0^2. Applying the Chain Rule on equation (4) and the Implicit Function Theorem to equation (2),

$$\frac{\partial}{\partial d_0^2}\left(\frac{\partial \log \hat{k}}{\partial r_1^1}\right) = \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} + \frac{\partial^2 \log k}{\partial r_1^1 \partial \bar{y}_1}\frac{d\bar{y}_1}{d d_0^2}.$$

From the analysis of the case in the main text, see Proposition 3, we know that we are only left with the terms involving $\partial y_1\left(r_1^1\right)/\partial r_1^1$,

$$\begin{aligned}
&\frac{\partial}{\partial d_0^2}\left(\frac{\partial \log \hat{k}}{\partial r_1^1}\right)\\
= \;& -\frac{1}{\omega^2\left(1+r_1^1\right)^2}\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\\
&+\frac{1}{\omega^2\left(1+r_1^1\right)}\frac{\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{r_1^1}\left\{\frac{\left(1+\tilde{r}_1^1\right)\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}\\
= \;& \frac{1}{\omega^2\left(1+r_1^1\right)}\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\frac{\left(1+r_0^2\right)}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(1+\tilde{r}_1^1\right)\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}\\
&\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\tilde{r}_1^1-r_1^1\right)}{y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2}\right.\\
&\left.\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2}\right\}<0.
\end{aligned}$$

where the inequality uses the first-order condition for the optimal maturity choice, i.e., equation (2), and the fact that the expression

$$\frac{\left(\tilde{r}_1^1-r_1^1\right)}{y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2}$$

is an increasing function of $\tilde{r}_1^1$. Similarly,

$$\begin{aligned}
\frac{\partial}{\partial d_0}\left(\frac{\partial \log \hat{k}}{\partial r_1^1}\right) =&\frac{\partial^2 \log k}{\partial r_1^1 \partial d_0}+\frac{\partial^2 \log k}{\partial r_1^1 \partial \bar{y}_1}\frac{d\bar{y}_1}{dd_0}\\
=&\frac{1+r_0^1}{1+r_1^1}\frac{1}{\omega^2}\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\\
&-\frac{1}{\left[\omega\left(r_1^1\right)\right]^2}\frac{\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\left(1+r_1^1\right)}\left(1+r_0^1\right)\\
=&0.
\end{aligned}$$

We next consider the case in which entrepreneurs are heterogeneous with respect to the initial leverage and the income in the final period y_2. Equation (2) defines implicitly a function relating y_2 and d_0 and d_0^2, which, abusing notation, we denote $y_2(d_0, d_0^2)$.

Using this notation, we can define the reduced form relationship between investment, the interest rate shock, leverage and debt maturity as

$$\hat{k}(r_1^1, d_0, d_0^2) = k(r_1^1, d_0, d_0^2, y_2(d_0, d_0^2)), \tag{5}$$

where we have omitted the dependence of k on parameters that are assumed to be common across entrepreneurs, i.e., y_1, r_0^1, r_0^2. Applying the Chain Rule on equation (5) and the Implicit Function Theorem to equation (2),

$$\begin{aligned}
\frac{\partial}{\partial d_0^2}\left(\frac{\partial \log \hat{k}}{\partial r_1^1}\right) =& \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_2}\frac{dy_2}{dd_0^2} \\
=& -\frac{1}{\omega^2\left(1+r_1^1\right)^2}\left(\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1} \\
&+\frac{1}{\omega^2}\frac{\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\left(1+r_1^1\right)^2} \\
&\frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}.
\end{aligned}$$

Rearranging

$$\begin{aligned}
=&\frac{1}{\omega^2}\frac{1}{(1+r_1^1)^2}\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\frac{(1+r_0^1)}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} \\
&\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\tilde{r}_1^1-r_1^1\right)\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\} \\
&<0.
\end{aligned}$$

where, as before, the last equality uses the first-order condition for the optimal maturity choice, i.e., equation (2), and the fact that the expression

$$\frac{\left(\tilde{r}_1^1-r_1^1\right)}{y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2}$$

is an increasing function of $\tilde{r}_1^1$. Similarly,

$$\begin{aligned}\frac{\partial}{\partial d_0}\left(\frac{\partial \log \hat{k}}{\partial r_1^1}\right) =& \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_2}\frac{dy_2}{dd_0} \\ =& \frac{1+r_0^1}{1+r_1^1}\frac{1}{\omega^2}\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1} \\ & - \frac{1}{\omega}\frac{\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\left(1+r_1^1\right)^2} \\ & \left(1+r_0^1\right)\frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(1+\tilde{r}_1^1\right)\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}.\end{aligned}$$

Rearranging

$$\begin{aligned}=& -\frac{\left(1+r_0^1\right)}{\omega^2}\frac{1}{(1+r_1^1)^2}\frac{\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} \\ & \mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\tilde{r}_1^1-r_1^1\right)\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\} \\ >& 0.\end{aligned}$$

Next, we analyse a counterpart of the result in Proposition 4. When entrepreneurs are heterogeneous with respect to the initial leverage and the term premium r_0^2, equation (2) defines implicitly a function relating r_0^2 and d_0 and d_0^2, which, abusing notation, we denote $r_0^2(d_0, d_0^2)$. Using this notation, we can define the reduced form relationship between investment, the interest rate shock, leverage and debt maturity as

$$\hat{k}(r_1^1, d_0, d_0^2) = k(r_1^1, d_0, d_0^2, r_0^2(d_0, d_0^2)), \tag{6}$$

where we have omitted the dependence of k on parameters that are assumed to be common across entrepreneurs, i.e., y_1, y_2, r_0^1. Applying the Chain Rule on equation

(6) and the Implicit Function Theorem to equation (2),

$$
\begin{aligned}
\frac{\partial}{\partial d_0^2}\left(\frac{\partial \log \hat{k}}{\partial r_1^1}\right) =& \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} + \frac{\partial^2 \log k}{\partial r_1^1 \partial r_0^2}\frac{dr_0^2}{dd_0^2} \\
=& \frac{\left(1+r_0^1\right)\left(1+r_0^2\right)}{\omega^2\left(1+r_1^1\right)^2}\left[\frac{y_2}{1+r_0^2}+\frac{y_1}{1+r_0^1}-d_0-\left(\frac{1+r_1^1}{1+r_0^2}-\frac{1}{\left(1+r_0^1\right)}\right)\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}\right] \\
&+ \left[\frac{1}{\omega^2}\frac{\left(y_2-(1+r_0^2)d_0^2\right)d_0^2}{(1+r_1^1)^2}-\frac{1}{\omega(r_1^1)}\frac{d_0^2}{(1+r_1^1)^2}\right] \\
&\frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}.
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
=& \frac{\left(1+r_0^1\right)}{\omega^2}\frac{y_2-\left(1+r_0^2\right)\left(d_0-\frac{y_1}{1+r_0^1}\right)-\frac{1}{1+r_0^1}\left(\left(1+r_0^1\right)\left(1+r_1^1\right)-\left(1+r_0^2\right)\right)\frac{\partial y_1\left(r_1^1\right)}{\partial r_1^1}}{(1+r_1^1)^2} \\
&- \frac{\left(1+r_1^1\right)}{\omega^2}\frac{\left(y_1-\left(1+r_0^1\right)\left(d_0-d_0^2\right)\right)d_0^2}{(1+r_1^1)^2} \\
&\frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)^2}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)}{\left[y_2+\left(1+\tilde{r}_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left(\left(1+\tilde{r}_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right)d_0^2\right]^2}\right\}} \\
=& \frac{\left(1+r_0^1\right)}{\omega^2}\frac{y_2-\left(1+r_0^2\right)\left(d_0-\frac{y_1}{1+r_0^1}\right)}{(1+r_1^1)^2} > 0,
\end{aligned}
$$

where the second equality uses the fact that $d_0^2 = d_0 - y_1/(1+r_0^1)$ when $1 + r_0^2 =$

$(1+r_0^1)\,\mathbb{E}\,(1+r_1^1)$, and the inequality follows from Assumption 2.a. Similarly,

$$
\begin{aligned}
\frac{\partial}{\partial d_0}\left(\frac{\partial \log \hat{k}}{\partial r_1^1}\right) =& \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0} + \frac{\partial^2 \log k}{\partial r_1^1 \partial r_0^2}\frac{dr_0^2}{dd_0} \\
=& -\frac{(1+r_0^1)}{\omega^2}\frac{y_2-(1+r_0^2)\,d_0^2}{(1+r_1^1)^2} \\
& -\left[\frac{1}{\omega^2}\frac{(y_2-(1+r_0^2)d_0^2)\,d_0^2}{(1+r_1^1)^2} - \frac{1}{\omega}\frac{d_0^2}{(1+r_1^1)^2}\right] \\
& \frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{(1+\tilde{r}_1^1)(1+r_0^1)((1+\tilde{r}_1^1)(1+r_0^1)-(1+r_0^2))}{[y_2+(1+\tilde{r}_1^1)(y_1-(1+r_0^1)d_0)+((1+\tilde{r}_1^1)(1+r_0^1)-(1+r_0^2))d_0^2]^2}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{y_2+(1+\tilde{r}_1^1)(y_1-(1+r_0^1)d_0)}{[y_2+(1+\tilde{r}_1^1)(y_1-(1+r_0^1)d_0)+((1+\tilde{r}_1^1)(1+r_0^1)-(1+r_0^2))d_0^2]^2}\right\}}.
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
=& -\frac{(1+r_0^1)}{\omega^2}\frac{y_2-(1+r_0^2)\,d_0^2}{(1+r_1^1)^2} \\
& +\frac{(1+r_1^1)}{\omega^2}\frac{(y_1-(1+r_0^1)\,(d_0-d_0^2))\,d_0^2}{(1+r_1^1)^2} \\
& \frac{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{(1+\tilde{r}_1^1)(1+r_0^1)((1+\tilde{r}_1^1)(1+r_0^1)-(1+r_0^2))}{[y_2+(1+\tilde{r}_1^1)(y_1-(1+r_0^1)d_0)+((1+\tilde{r}_1^1)(1+r_0^1)-(1+r_0^2))d_0^2]^2}\right\}}{\mathbb{E}_{\tilde{r}_1^1}\left\{\frac{y_2+(1+\tilde{r}_1^1)(y_1-(1+r_0^1)d_0)}{[y_2+(1+\tilde{r}_1^1)(y_1-(1+r_0^1)d_0)+((1+\tilde{r}_1^1)(1+r_0^1)-(1+r_0^2))d_0^2]^2}\right\}} \\
=& -\frac{(1+r_0^1)}{\omega^2}\frac{y_2-(1+r_0^2)\,d_0^2}{(1+r_1^1)^2} < 0.
\end{aligned}
$$

where the second equality uses the fact that $d_0^2 = d_0 - y_1/(1+r_0^1)$ when $1+r_0^2 = (1+r_0^1)\,\mathbb{E}\,(1+r_1^1)$, and the inequality follows from the condition $y_2-(1+r_0^2)d_0^2$.

E Limited Commitment, Risk Neutrality, and Diminishing Returns

In this appendix we consider an alternative set of assumptions that are also common in the macro finance literature: limited commitment, risk neutrality, and diminishing returns (Khan and Thomas, 2013; Buera et al., 2015). We modify the benchmark model by introducing these assumptions sequentially. In addition, we allow for interior values for the depreciation of investment $\delta \in (0,1]$ (the benchmark model in the main text corresponds to the case $\delta = 1$).

First, we consider the case in which the financial friction is limited commitment instead of uninsurable investment risk. In particular, in the benchmark case with a constant return technology in the interim period, the investment and leverage decisions in the interim period must satisfy the following limited commitment constraint

$$\begin{aligned} & y_2 + (z+1-\delta)\,k - \left(1+r_0^2\right) d_0^2 - \left(1+r_1^1\right)\left(k + \left(1+r_0^1\right)\left(d_0 - d_0^2\right) - y_1\right) \\ \geq & \left(1-\phi_y\right) y_2 + \left(1-\phi_z\right) zk + \left(1-\phi_k\right)(1-\delta)\,k. \end{aligned}$$

The left hand side is the income in the case that the short and long term debts are repaid, while the right hand side is the income in the case of default. The parameters ϕ_y, ϕ_z, and ϕ_k are the fractions of long-term cash flows, the return to the investment in the interim period, and the undepreciated capital, respectively, that can be recovered in the event of default. In this appendix we assume that the return to the investment in the interim period is deterministic. The only uncertainty, from the point of view of the initial period, is about the realisation of the interest rate in the interim period.

In the following subsection we consider three different cases: (i) risk aversion and constant returns; (ii) risk neutrality and constant returns; (iii) risk neutrality and diminishing returns.

E.1 Risk Aversion and Constant Returns

$$\max_{d_0^2} E_{r_1^1} \max_{k,d_1^1} \log\left[y_2 + (z+1-\delta)\,k - \left(1+r_1^1\right) d_1^1 - \left(1+r_0^2\right) d_0^2\right]$$

s.t.

$$d_1^1 = k + \left(1+r_0^1\right)\left(d_0 - d_0^2\right) - y_1,$$

and

$$\begin{aligned} & y_2 + (z+1-\delta)\,k - \left(1+r_0^2\right) d_0^2 - \left(1+r_1^1\right) d_1^1 \\ \geq & \left(1-\phi_y\right) y_2 + \left(1-\phi_z\right) zk + \left(1-\phi_k\right)(1-\delta)\,k, \end{aligned}$$

or

$$\max_{d_0^2} E_{r_1^1} \quad \{\max_k \log \quad \left[y_2 + \left(z - \delta - r_1^1\right) k - \left(1 + r_1^1\right) \left(\left(1 + r_0^1\right) d_0 - y_1\right) \right.$$
$$\left. - \left(\left(1 + r_1^1\right)\left(1 + r_0^1\right) - \left(1 + r_0^2\right)\right) d_0^2 \right] \}$$

s.t.

$$\left(1 + r_0^2\right) d_0^2 + \left(1 + r_1^1\right)\left(k + \left(1 + r_0^1\right)\left(d_0 - d_0^2\right) - y_1\right) \leq \phi_y y_2 + \phi_z z k + \phi_k \left(1 - \delta\right) k.$$

Under the assumption that the constraint is binding for all possible realizations of the interest rate in the interim period, i.e., $1 + \underline{r} - \phi_z z - \phi_k (1 - \delta) > 0$, a simple upper bound on the capital choice

$$k \leq \frac{\phi_y y_2 + (1 + r_1^1)(y_1 + (1 + r_0^1)(a + d_0^2)) - (1 + r_0^2) d_0^2}{1 + r_1^1 - \phi_z z - \phi_k (1 - \delta)}, \text{ all } r_1^1.$$

The following condition guarantees that investment is positive for all realizations of the interest rate in the interim period r_1^1,

$$z - \bar{r} - \delta > 0.$$

If $\phi_y = 1$, we obtain the same solution as in the benchmark case with uninsurable investment risk and no-collateral constraint. In particular, the investment decision is a linear function of the net worth at the beginning of the interim period,

$$k = \bar{k}\left(r_1^1\right) \underbrace{\left[\frac{y_2}{1 + r_1^1} + y_1 - \left(1 + r_0^1\right) d_0 + \left(1 + r_0^1 - \frac{1 + r_0^2}{1 + r_1^1}\right) d_0^2\right]}_{\omega}$$

where

$$\bar{k}\left(r_1^1\right) = \frac{1 + r_1^1}{1 + r_1^1 - \phi_z z - \phi_k (1 - \delta)}$$

is a strictly decreasing function of r_1^1 provided $\phi_z z + \phi_k (1 - \delta) > 0$ and $1 + r_1^1 - \phi_z z - \phi_k (1 - \delta) > 0$, and consumption in the final period equals

$$c_2 = \bar{c}_2\left(r_1^1\right)\left[y_2 + \left(1 + r_1^1\right)\left(y_1 - \left(1 + r_0^1\right) d_0\right) + \left[\left(1 + r_1^1\right)\left(1 + r_0^1\right) - \left(1 + r_0^2\right)\right] d_0^2\right]$$

where

$$\bar{c}_2\left(r_1^1\right) = \frac{z-\delta-r_1^1}{1+r_1^1}\bar{k}\left(r_1^1\right)+1$$
$$= \frac{(1-\phi_z)\,z+(1-\delta)\,(1-\phi_k)}{1+r_1^1-\phi_z z-\phi_k\,(1-\delta)},$$

a decreasing function of r_1^1.

Conditional on the value of the net worth, the investment is a decreasing function of the interest rate. As in the benchmark case, the optimal maturity choice solves

$$\max_{d_0^2} E_{r_1^1} \log\left\{\bar{c}_2\left(r_1^1\right)\left[y_2+\left(1+r_1^1\right)\left(y_1-\left(1+r_0^1\right)d_0\right)+\left[\left(1+r_1^1\right)\left(1+r_0^1\right)-\left(1+r_0^2\right)\right]d_0^2\right]\right\}$$

or

$$\max_{d_0^2} E_{r_1^1} \log\ \left[y_2+(1+r_1^1)\,(y_1-(1+r_0^1)\,d_0)+[(1+r_1^1)\,(1+r_0^1)-(1+r_0^2)]\,d_0^2\right]\ .$$

Clearly, all the results in the main text follow for this specification.

E.2 Risk Neutrality and Constant Returns

We next consider the case with risk neutrality. In this case, the maturity choice solves

$$\max_{d_0^2} E_{r_1^1}\left\{\bar{c}_2\left(r_1^1\right)\left[\ y_2+(1+r_1^1)\,(y_1-(1+r_0^1)\,d_0)+[(1+r_1^1)\,(1+r_0^1)-(1+r_0^2)]\,d_0^2\ \right]\right\}$$
$$=[(1-\phi_z)\,z+(1-\phi_k)\,(1-\delta)]\,E_{r_1^1}\left\{\ \frac{y_2+\left(1+r_1^1\right)\left(y_1+\left(1+r_0^1\right)a\right)}{1+r_1^1-\phi_z z-\phi_k(1-\delta)}\ \right\}$$
$$+[(1-\phi_z)\,z+(1-\phi_k)\,(1-\delta)]\max_{d_0^2} E_{r_1^1}\left\{\frac{(1+r_1^1)\,(1+r_0^1)-(1+r_0^2)}{1+r_1^1-\phi_z z-\phi_k\,(1-\delta)}\right\}d_0^2$$

The optimal maturity choice takes extremes values depending on the sign of

$$E_{r_1^1}\left\{\frac{(1+r_1^1)\,(1+r_0^1)-(1+r_0^2)}{1+r_1^1-\phi_z z-\phi_k\,(1-\delta)}\right\}.$$

For instance, if the expectation hypothesis holds, i.e., $E_{r_1^1}\left[(1+r_1^1)\,(1+r_0^1)-(1+r_0^2)\right]=0$, then the optimal choice is to set d_0^2 to $-\infty$. In this case, by buying (an arbitrarily large quantity of) long-term bonds the entrepreneur transfer an arbitrarily large amount of resources to the low interest rate states, in which leverage, $(1+r_1^1)\,/\,(1+r_1^1-\phi_z z-\phi_k\,(1-\delta)$ is the largest.

E.3 Risk Neutrality and Diminishing Returns

Next, to obtain an interior solution to the optimal maturity choice we introduce diminishing returns to capital. In particular, we assume that the return to the investment in the interim period is given by the following Cobb-Douglas technology

$$y = zk^{\alpha}.$$

In addition, to obtain a closed form solution for the constrained level of investment we assume $\phi_z = 0$. Given this assumption, the limited commitment constraint is given by

$$\begin{aligned}&\phi_y y_2 + zk^{\alpha} + (1-\delta)\,k - \left(1+r_0^2\right) d_0^2 - \left(1+r_1^1\right)\left(k - \left(1+r_0^1\right)\left(-d_0 + d_0^2\right) - y_1\right)\\ \geq\, & zk^{\alpha} + (1-\phi_k)\,(1-\delta)\,k\end{aligned}$$

or

$$\begin{aligned} k &\leq k^c\left(d_0^2, r_1^1\right) \\ &\equiv \frac{1+r_1^1}{1+r_1^1-\phi_k\,(1-\delta)}\left[\frac{\phi_y y_2 - (1+r_0^2)\,d_0^2}{1+r_1^1} + \left(1+r_0^1\right)\left(-d_0+d_0^2\right) + y_1\right].\end{aligned}$$

Furthermore, to simplify the exposition we set $\phi_y = 1$ and $\delta = 1$, which leads to

$$\begin{aligned} k &\leq \frac{y_2 - (1+r_0^2)\,d_0^2}{1+r_1^1} + y_1 - \left(1+r_0^1\right)\left(d_0 - d_0^2\right). \\ &= \omega. \end{aligned} \tag{7}$$

Under the simplifying assumption entrepreneurs can only pledge the cash flow of the long-term project in the last period and, therefore, the maximum feasible investment is given by the value of the net-worth in the interim period. We consider cases in which there is a need to refinance debts in the interim period, i.e., $y_1 - (1+r_0^1)\,(d_0 - d_0^2) < 0$. Given this restriction, there need to be a strictly positive cash flow in the last period, i.e., $y_2 - (1+r_0^2)\,d_0^2 > 0$, for strictly positive investment to be incentive compatible. In addition, this assumption implies that the upper bound on the investment in the interim period is a decreasing function of the interest rate in the interim period and, therefore, there is an upper bound on the realisation of the interest rate $\hat{r}$ consistent with a positive investment. The upper bound $\hat{r}$ is given by

$$\hat{r} = \frac{y_2 - (1 + r_0^2)\, d_0^2}{(1 + r_0^1)\,(d_0 - d_0^2) - y_1} - 1,$$

with

$$\begin{aligned}
\frac{\partial \hat{r}}{\partial d_0^2} &= -\frac{(1 + r_0^2)}{(1 + r_0^1)\,(d_0 - d_0^2) - y_1} + \frac{(1 + r_0^1)\,(y_2 - (1 + r_0^2)\, d_0^2)}{\left[(1 + r_0^1)\,(d_0 - d_0^2) - y_1\right]^2} \\
&= \left(1 + r_0^2\right)\left(1 + r_0^1\right) \frac{\frac{y_1}{1+r_0^1} + \frac{y_2}{1+r_0^2} - d_0}{\left[(1 + r_0^1)\,(d_0 - d_0^2) - y_1\right]^2} > 0
\end{aligned}$$

where the inequality follows from assumptions 1 and 2.

As the amount of long-term debt changes, the debt limit will pivot around a value of the interest rate in the interim period $\tilde{r}$ that makes the realised return of long-term debt zero, i.e.,

$$1 + \tilde{r} = \frac{1 + r_0^2}{1 + r_0^1}.$$

For values of the interest rate in the interim period lower than this, $r_1^1 < \tilde{r}$, the debt limit becomes tighter (looser) when amount of long-term debt increases (decreases). The converse is true for values of the interest rate higher than this one, $r_1 > \tilde{r}$.

The unconstrained level of investment equals

$$k^u\left(r_1^1\right) = \left(\frac{\alpha z}{1 + r_1^1}\right)^{\frac{1}{1-\alpha}}. \tag{8}$$

Thus, the capital invested by an individual entrepreneur is given by the minimum of (7) and (8), i.e.,

$$k\left(r_1^1, d_0^2\right) = \min\left\{k^c\left(r_1^1, d_0^2\right), k^u\left(r_1^1\right)\right\},$$

where, for simplicity, we have omitted the dependence of investment on the other variables in the model, i.e., d_0, y_1, y_2,...

If the entrepreneur is constrained, i.e., $k^c\left(d_0^2, r_1^1\right) < k^u\left(r_1^1\right)$, the sensitivity of investment to the interest rate shock in the interim period equals

$$\frac{\partial \log k}{\partial r_1^1} = -\frac{1}{\omega} \frac{y_2 - (1 + r_0^2)\, d_0^2}{\left(1 + r_1^1\right)^2}.$$

This expression is equal to the second term in equation (7) in the main text. In the present model the first term in equation (7) is zero, as $\bar{k}\left(r_1^1\right) = 1$, a value independent of r_1^1. In addition, in the present model the sensitivity of investment to the interest rate shock is zero for unconstrained individuals, i.e., $k^c\left(d_0^2, r_1^1\right) < k^u\left(r_1^1\right)$. Therefore, we obtain the same expressions for effect of an exogenous change in leverage and debt maturity on the sensitivity of investment to the interest rate shock

$$\frac{\partial^2 \log k}{\partial r_1^1 \partial d_0} = -\frac{\left(1 + r_0^1\right)}{\left(1 + r_1^1\right)^2} \frac{1}{\omega^2} \left[y_2 - \left(1 + r_0^2\right) d_0^2\right]$$

and

$$\frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} = \frac{\left(1 + r_0^2\right)\left(1 + r_0^1\right)}{\left(1 + r_1^1\right)^2} \frac{1}{\omega^2} \left[\frac{y_2}{1 + r_0^2} + \frac{y_1}{1 + r_0^1} - d_0\right] > 0.$$

These expressions are equal to the ones obtained for the model in the main text. Therefore, exact counterparts to Propositions 1 and 2 in the main text are true in this extension provided that we are considering a value of r_1^1 for which the entrepreneur is constrained, i.e., $k^c\left(r_1^1, d_0^2\right) < k^u\left(r_1^1\right)$.

We also use below the follow to cross partial derivatives

$$\frac{\partial^2 \log k}{\partial r_1^1 \partial y_1} = \frac{1}{\omega^2} \frac{y_2 - (1 + r_0^2)\, d_0^2}{\left(1 + r_1^1\right)^2} > 0.$$

and

$$\begin{aligned}
\frac{\partial^2 \log k}{\partial r_1^1 \partial y_2} &= -\frac{\frac{1}{\left(1+r_1^1\right)^2}}{y_1 - (1 + r_0^1)\, d_0 + \frac{y_2}{1+r_1^1} + \left(1 + r_0^1 - \frac{1+r_0^2}{1+r_1^1}\right) d_0^2} \\
&\quad + \frac{\frac{1}{1+r_1^1} \frac{y_2 - \left(1+r_0^2\right) d_0^2}{\left(1+r_1^1\right)^2}}{\left[y_1 - \left(1 + r_0^1\right) d_0 + \frac{y_2}{1+r_1^1} + \left(1 + r_0^1 - \frac{1+r_0^2}{1+r_1^1}\right) d_0^2\right]^2} \\
&= \frac{1}{\left(1 + r_1^1\right)^2} \frac{1}{\omega^2} \left[\left(1 + r_0^1\right)\left(d_0 - d_0^2\right) - y_1\right] > 0.
\end{aligned}$$

E.3.1 Maturity Choice

In order to have a clean example where long-term debt is optimal we assume that the entrepreneur is just constrained when she issues no long-term debt, i.e.,

$$\left(\frac{\alpha z}{1+\tilde{r}}\right)^{\frac{1}{1-\alpha}} = y_1 - \left(1+r_0^1\right) d_0 + \frac{y_2}{1+\tilde{r}}, \tag{9}$$

and that is becomes strictly constrained (unconstrained) to the right (left) of $\tilde{r}$, i.e.,

$$-\frac{(\alpha z)^{\frac{1}{1-\alpha}}}{1-\alpha}\left(\frac{1}{1+\tilde{r}}\right)^{\frac{1}{1-\alpha}-1} > -\frac{y_2}{(1+\tilde{r})^2}. \tag{10}$$

Alternatively, we can write these conditions as

$$\frac{1}{1-\alpha}\left(\frac{\alpha z}{1+\tilde{r}}\right)^{\frac{1}{1-\alpha}} = \frac{1}{1-\alpha}\left[y_1 - \left(1+r_0^1\right) d_0 + \frac{y_2}{1+\tilde{r}}\right] < \frac{y_2}{1+\tilde{r}},$$

requiring that the financial needs in the interim period are sufficiently large, $\left(1+r_0^1\right) d_0 - y_1 > y_2\alpha/\left(1-\alpha\right)$, and that productivity of the investment takes a particular value.

The maturity choice solves

$$\max_{d_0^2} \int_{\underline{r}}^{\bar{r}} \left\{\left[\left(1+r_0^1\right)(1+r) - \left(1+r_0^2\right)\right] d_0^2 + z\left[k\left(r, d_0^2\right)\right]^{\alpha} - (1+r)\, k\left(r, d_0^2\right)\right\} dF(r)$$

If the expectation hypothesis hold, i.e., $\int_{\underline{r}}^{\bar{r}} \left[\left(1+r_0^1\right)(1+r) - \left(1+r_0^2\right)\right] dF(r) = 0$, the optimal maturity choice solves the following simplified problem

$$\max_{d_0^2} \int_{\underline{r}}^{\bar{r}} \left[z\left[k\left(r, d_0^2\right)\right]^{\alpha} - (1+r)\, k\left(r, d_0^2\right)\right] dF(r).$$

The first order condition of this problem is

$$\int_{r: k^c\left(r, d_0^2\right) < k^u(r)} \left[\alpha z\left[k\left(r, d_0^2\right)\right]^{\alpha-1} - (1+r)\right]\left(1 + r_0^1 - \frac{1+r_0^2}{1+r}\right) dF(r) = 0$$

or, using the assumption that the expectation hypothesis holds, this condition can be rewritten as

$$\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha z \left[k\left(r,d_0^2\right)\right]^{\alpha-1}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) dF(r)=0. \tag{11}$$

Given conditions (9) and (10), and provided the range of interest rate is small enough, i.e., $\bar{r}-\underline{r}$ is small enough and $\tilde{r}\in(\underline{r},\bar{r})$, it follows from the first order condition that it is optimal to issue long-term debt. That is, it is optimal to transfer resources from state of the world with low interest rate when the entrepreneur is unconstrained to high interest rate states when the entrepreneur is constrained.[6]

Applying the Implicit Function Theorem to (11), it follows that the optimal maturity choice is a decreasing function of the long-term interest rate in the neighbourhood of $d_0^2=0$,

$$\begin{aligned}
\left.\frac{\partial d_0^2}{\partial r_0^2}\right|_{d_0^2=0} &= \frac{1}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha(1-\alpha) z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)^2 dF(r)} \\
&\quad \left[d_0^2 \int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \frac{\alpha-1}{1+r}\alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) dF(r)\right. \\
&\quad \left. -\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-1} dF(r)\right], \\
&= -\frac{1}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha(1-\alpha) z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)^2 dF(r)} \\
&\quad \int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-1} dF(r)<0,
\end{aligned}$$

where the second equality uses that we are evaluating the derivative at $d_0^2=0$.

Similarly, we have that the amount of long-term debt is a decreasing function of

[6] For this result we require that $\underline{r}$ is not too low. For sufficiently low values of $\underline{r}$, the unconstrained level of capital in (8) will be arbitrarily larger than the amount feasible given by (7). In this case, the the marginal value of resources in the low interest rate states will be particularly high, and choosing a negative value for the long term debt (positive amount of long-term assets) might be optimal.

the cash flow of the long-term project in the first and second periods,

$$\begin{aligned}
\frac{\partial d_0^2}{\partial y_1} &= -\frac{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right)}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)^2 dF\left(r\right)} \\
&= -\frac{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \frac{1}{k\left(r,d_0^2\right)}\alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-1}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right)}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)^2 dF\left(r\right)} \\
&< 0, \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial d_0^2}{\partial y_2} &= -\frac{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\frac{1}{1+r}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right)}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)^2 dF\left(r\right)} \\
&= -\frac{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \frac{1}{k\left(r,d_0^2\right)(1+r)}\alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-1}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right)}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)^2 dF\left(r\right)} \\
&< 0, \qquad (13)
\end{aligned}$$

where the inequality in both cases follow from the first order condition (11) and the fact that $k^c\left(r,d_0^2\right)$ and $k^c\left(r,d_0^2\right)\left(1+r\right)$ are a strictly decreasing function of r, given that we assume $y_2-\left(1+r_0^2\right)d_0^2>0$ and $y_1-\left(1+r_0^1\right)\left(d_0-d_0^2\right)<0$.

Finally, we are ready to characterise how the reduced form sensitivity of investment with respect to the interest rate is affected by leverage and maturity. We first look at the case in which entrepreneurs are heterogeneous with respect to the initial leverage and the income in the interim period, y_1. In this case, the reduced form relationship between investment, leverage and maturity is given by

$$\hat{k}\left(r_1^1,d_0,d_0^2\right)=k\left(r_1^1,d_0,d_0^2,y_1\left(d_0,d_0^2\right)\right), \qquad (14)$$

where the $y_1\left(d_0,d_0^2\right)$ is implicitly defined by equation (11).

Applying the Chain Rule on equation (14) and using (12),

$$\begin{aligned}
\frac{\partial}{\partial d_0^2}\frac{\partial \log \hat{k}}{\partial r_1^1} &= \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_1}\frac{\partial y_1}{\partial d_0^2} \\
&= \frac{1}{\omega^2}\frac{\left(1+r_0^2\right)\left(1+r_0^1\right)}{\left(1+r_1^1\right)^2}\left[\frac{y_2}{1+r_0^2}+\frac{y_1}{1+r_0^1}-d_0\right] \\
&\quad -\frac{1}{\omega^2}\frac{y_2-\left(1+r_0^2\right)d_0^2}{\left(1+r_1^1\right)^2} \\
&\quad \frac{\int_{r:k^c\left(r,d_0^2\right)>k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)^2 dF\left(r\right)}{\int_{r:k^c\left(r,d_0^2\right)>k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) dF\left(r\right)},
\end{aligned}$$

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$$\begin{aligned}
&= \frac{\left(1+r_0^2\right)}{\left(1+r_1^1\right)^2}\frac{1}{\omega^2}\frac{1}{\int_{r:k^c\left(r,d_0^2\right)>k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) dF\left(r\right)} \\
&\quad \int_{r:k^c\left(r,d_0^2\right)>k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) \\
&\quad \left[\left(1+r_0^2\right)\left(1+r_0^1\right)\left[\frac{y_2}{1+r_0^2}+\frac{y_1}{1+r_0^1}-d_0\right]\right. \\
&\quad \left.-\left(y_2-\left(1+r_0^2\right)d_0^2\right)\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)\right]dF\left(r\right) \\
&= \frac{\left(1+r_0^2\right)}{\left(1+r_1^1\right)^2}\frac{1}{\omega^2}\frac{1}{\int_{r:k^c\left(r,d_0^2\right)>k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) dF\left(r\right)} \\
&\quad \int_{r:k^c\left(r,d_0^2\right)>k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-1}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) \\
&= 0,
\end{aligned}$$

where the last equality follows from (11).

Again, applying the Chain Rule on equation (14) and the Implicit Function Theorem to equation (11) we obtain:

$$\begin{aligned}
\frac{\partial}{\partial d_0}\frac{\partial \log \hat{k}}{\partial r_1^1} &= \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_1}\frac{\partial y_1}{\partial d_0} \\
&= -\frac{\left(1+r_0^1\right)}{\left(1+r_1^1\right)^2}\frac{1}{\omega^2}\left[y_2 - \left(1+r_0^2\right)d_0^2\right] \\
&\quad + \frac{1}{\omega^2}\frac{y_2 - \left(1+r_0^2\right)d_0^2}{\left(1+r_1^1\right)^2}\left(1+r_0^1\right) \\
&= 0.
\end{aligned}$$

Finally, we study the case in which entrepreneurs are heterogeneous with respect to the initial leverage and the income in the last period, y_2. In this case, the reduced form relationship between investment, leverage and maturity is given by

$$\hat{k}\left(r_1^1, d_0, d_0^2\right) = k\left(r_1^1, d_0, d_0^2, y_2\left(d_0, d_0^2\right)\right),$$

where the $y_2\left(d_0, d_0^2\right)$ is implicity defined by equation (11).

Applying the Chain Rule on equation (14) and using (13),

$$\begin{aligned}
\frac{\partial^2 \log \hat{k}}{\partial r_1^1 \partial d_0^2} &= \frac{\partial^2 \log k}{\partial r_1^1 \partial d_0^2} + \frac{\partial^2 \log k}{\partial r_1^1 \partial y_2}\frac{\partial y_2}{\partial d_0^2} \\
&= \frac{\left(1+r_0^1\right)}{\omega^2}\frac{y_2 - \left(1+r_0^2\right)\left(d_0 - \frac{y_1}{1+r_0^1}\right)}{\left(1+r_1^1\right)^2} \\
&\quad - \left[\frac{1}{\omega^2}\frac{y_2 - (1+r_0^2)d_0^2}{(1+r_1^1)^2}\frac{1}{1+r_1^1} - \frac{1}{\omega}\frac{1}{(1+r_1^1)^2}\right] \\
&\quad \frac{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha\left(\alpha-1\right) z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1 - \frac{1+r_0^2}{1+r}\right)^2 dF\left(r\right)}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)} \alpha\left(\alpha-1\right) z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\frac{1}{1+r}\left(1+r_0^1 - \frac{1+r_0^2}{1+r}\right) dF\left(r\right)},
\end{aligned}$$

rearrenging

$$
\begin{aligned}
&= \frac{1}{\omega^2}\frac{1}{(1+r_1^1)^2}\frac{1+r_0^1}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\frac{1}{1+r}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right)} \\
&\quad \int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) \\
&\quad \left[\frac{y_2}{1+r}+y_1-\left(1+r_0^1\right)d_0+\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)d_0^2\right]dF\left(r\right) \\
&= \frac{1}{\omega^2}\frac{1}{(1+r_1^1)^2}\frac{1+r_0^1}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\frac{1}{1+r}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right)} \\
&\quad \left(\alpha-1\right)\int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha z\left[k\left(r,d_0^2\right)\right]^{\alpha-1}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right) \\
&= 0,
\end{aligned}
$$

where the last equality follows from (11).

Again, applying the Chain Rule on equation (14) and using (12),

$$
\begin{aligned}
\frac{\partial^2\log\hat{k}}{\partial r_1^1\partial d_0} &= \frac{\partial^2\log k}{\partial r_1^1\partial d_0}+\frac{\partial^2\log k}{\partial r_1^1\partial y_2}\frac{\partial y_2}{\partial d_0} \\
&= -\frac{\left(1+r_0^1\right)}{\omega^2}\frac{y_2-\left(1+r_0^2\right)d_0^2}{(1+r_1^1)^2} \\
&\quad +\left[\frac{1}{\omega^2}\frac{y_2-(1+r_0^2)d_0^2}{(1+r_1^1)^2}\frac{1}{1+r_1^1}-\frac{1}{\omega}\frac{1}{(1+r_1^1)^2}\right] \\
&\quad \frac{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)\left(1+r_0^1\right)dF\left(r\right)}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)\frac{1}{1+r}dF\left(r\right)},
\end{aligned}
$$

rearrenging,

$$
\begin{aligned}
&= -\frac{1}{\omega^2}\frac{1}{(1+r_1^1)^2}\frac{1+r_0^1}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)\frac{1}{1+r}dF\left(r\right)} \\
&\quad \int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right) \\
&\quad \left[\frac{y_2}{1+r}+y_1-\left(1+r_0^1\right)d_0+\left(1+r_0^1-\frac{1+r_0^2}{1+r}d_0^2\right)d_0^2\right]dF\left(r\right) \\
&= -\frac{1}{\omega^2}\frac{1}{(1+r_1^1)^2}\frac{1+r_0^1}{\int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-2}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)\frac{1}{1+r}dF\left(r\right)} \\
&\quad \int_{r:k^c\left(r,d_0^2\right)<k^u(r)}\alpha\left(\alpha-1\right)z\left[k\left(r,d_0^2\right)\right]^{\alpha-1}\left(1+r_0^1-\frac{1+r_0^2}{1+r}\right)dF\left(r\right) \\
&= 0,
\end{aligned}
$$

where the last equality follows from (11).

It is straightforward to show that a counterpart of Proposition 4 holds for this economy, provided that the optimal amount of long-term debt is a decreasing function of its cost, i.e, r_0^2. Thus, all the results in the benchmark model extends to this economy with risk neutral investors and diminishing returns to the investment in the interim period.

F Endogenous Leverage

In this appendix we consider a version of the model where initial leverage is endogenously determined by firms' productivity and initial resources in the first period. We perform the analysis in a model featuring diminishing returns and collateral constraints, extended to have a short-term investment decision in the first period, but abstracting from the maturity choice to focus on the endogenous determination of leverage. The analysis in this case is done for the case without cash flows from a pre-existing long-term projects. Given this assumptions, the constrained level of investment is independent of the realisation of the interest rate. This allows us to focus on the role of the different determinants of leverage, i.e., the initial net-worth and the productivity in the initial period.

As in the benchmark model, we consider the investment problem of an entrepreneur that lives for three periods, $t = 0, 1, 2$, facing investment opportunities in the first two periods and consuming in the last one. Entrepreneurs are heterogeneous with respect to the initial net worth a_0 and the productivity of their investment opportunities z_t, $t = 0, 1$. To simplify the exposition, we assume that the productivities are known to individuals at the beginning of period 0, and are distributed across agents according to

$$(z_0, z_1) \sim G_0(z_0) \, G_1(z_1) \, .$$

That is, productivities are assumed to be independent over time.

We model the investment opportunities of entrepreneurs as simple Cobb-Douglas technologies

$$z_t k_t^{\alpha}, \; t = 0, 1.$$

We abstract from uninsurable investment risk and, instead, we assume that investment is constrained by individual's net worth

$$k_t \leq \lambda a_t,$$

where λ parametrises the collateral constraint and a_t denotes the net worth at time $t = 0, 1$.

To simplify the analysis, we assume that entrepreneurs are risk neutral and do not discount the future. Therefore, there is no role for the maturity of debt. We therefore restrict the analysis to one period debt. Given this, we denote by r_t the one period interest rate.

Capital input choices solve

$$\max_{k_0, k_1} z_1 k_1^{\alpha} + (1 - \delta) \, k_1 - (1 + r_1) \, (k_1 - a_1)$$

$$s.t.$$

$$a_1 = z_0 k_0^{\alpha} + (1 - \delta) \, k_0 - (1 + r_0) \, (k_0 - a_0)$$

$$k_t \leq \lambda a_t, \; t = 0, 1.$$

The capital input at time $t = 0$

$$k_0 = \begin{cases} \lambda a_0 & \text{if } \lambda a_0 < \left(\frac{\alpha z_0}{r_0+\delta}\right)^{\frac{1}{1-\alpha}} \\ \left(\frac{\alpha z_0}{r_0+\delta}\right)^{\frac{1}{1-\alpha}} & \text{otherwise.} \end{cases} \tag{15}$$

The net worth at the beginning of the period $t = 1$

$$a_1 = \begin{cases} z_0 (\lambda a_0)^{\alpha} & \\ + \left[(1-\delta)\lambda - (1+r_0)(\lambda - 1)\right] a_0 & \text{if } \lambda a_0 < \left(\frac{\alpha z_0}{r_0+\delta}\right)^{\frac{1}{1-\alpha}} \\ (1-\alpha)\left(\frac{\alpha}{r_0+\delta}\right)^{\frac{\alpha}{1-\alpha}} z_0^{\frac{1}{1-\alpha}} + (1+r_0) a_0 & \text{otherwise .} \end{cases} \tag{16}$$

The initial leverage is

$$\begin{aligned} l_0 &= \frac{\max\{k_0 - a_0, 0\}}{k_0} \\ &= \max\left\{1 - \frac{a_0}{k_0}, 0\right\}. \end{aligned}$$

Using (15), we can express leverage as a function of initial net-worth and initial productivity

$$l_0 = \begin{cases} 1 - \frac{1}{\lambda} & \text{if } \lambda a_0 < \left(\frac{\alpha z_0}{r_0+\delta}\right)^{\frac{1}{1-\alpha}} \\ 1 - \frac{a_0}{\left(\frac{\alpha z_0}{r_0+\delta}\right)^{\frac{1}{1-\alpha}}} & \text{otherwise.} \end{cases} \tag{17}$$

If the capital input is constrained, then leverage is highest (and independent of initial net worth and productivity). Otherwise, leverage is a strictly decreasing function of the initial net worth and a strictly increasing function of the initial productivity.

The capital input at $t = 1$

$$k_1 = \begin{cases} \lambda a_1 & \text{if } \lambda a_1 < \left(\frac{\alpha z_1}{r_1+\delta}\right)^{\frac{1}{1-\alpha}} \\ \left(\frac{\alpha z_1}{r_1+\delta}\right)^{\frac{1}{1-\alpha}} & \text{otherwise.} \end{cases} \tag{18}$$

Using (16) and (18), we can write the average investment in the interim period of individuals with initial net worth a_0 and initial productivity z_0 as a function of the interest rate in period $t = 1$

$$k_1(r_1, a_0, z_0) = \int_0^{z_1^*} \left(\frac{\alpha z_1}{r_1 + \delta}\right)^{\frac{1}{1-\alpha}} dG_1(z) + (1 - G_1(\hat{z}_1))\, \lambda a_1(a_0, z_0)$$

where z^* is the productivity of the marginal entrepreneur who is unconstrained in the intermediate period

$$\lambda a_1(a_0, z_0) = \left(\frac{\alpha z_1^*}{r_1 + \delta}\right)^{\frac{1}{1-\alpha}}, \tag{19}$$

and the function $a_1(a_0, z_0)$ is defined in (16) (we omit the initial interest rate r_0 as an input of the interim investment and net worth functions).

The sensitivity of average investment to the interest rate in the interim period

$$\frac{\partial k_1(r_1, a_0, z_0)}{\partial r_1} = -\frac{1}{1-\alpha}(r_1 + \delta)^{-\frac{1}{1-\alpha}-1} \int_0^{\hat{z}_1} (\alpha z_1)^{\frac{1}{1-\alpha}} dG_1(z).$$

A change in the interest rate affects only the entrepreneurs whose investment is unconstrained, that is, entrepreneurs with relatively low productivity at time $t = 1$, i.e., $z_1 \leq z_1^*$.

As in the analysis in the main text, we are interested in characterising the reduced form relationship between investment, the interest rate (financial) shock, and initial leverage, which are the key variables in our empirical analysis.

To obtain a simple characterisation of this reduced form relationship, we assume that the initial heterogeneity is one-dimensional. We consider two polar cases: (i) z_0 is common and, therefore, entrepreneurs are heterogeneous only in terms of their initial net worth a_0 ; (ii) a_0 is common and, therefore, entrepreneurs are only heterogeneous in terms of their initial productivity z_0. In these cases, the reduced form relationship between investment, the interest rate shock, and initial leverage is

$$\hat{k}(r_1, l_0) = k(r_1, a_0(l_0), z_0)$$

or

$$\hat{k}(r_1, l_0) = k(r_1, a_0, z_0(l_0)),$$

depending on whether the heterogeneity stems from the initial net worth or the initial productivity, respectively. The relationships between the initial net worth or the initial productivity and leverage, $a_0(l_0)$ or $z_0(l_0)$, are derived from (17). To guarantee that leverage is interior, $l_0 \in (0, 1 - 1/\lambda)$, we focus on cases in which entrepreneurs are unconstrained in the first period.

Heterogeneous a_0, Common z_0

When the heterogeneity is solely in terms of the initial net worth a_0, the reduced form relationship between investment, the interest rate shock, and initial leverage is

$$\begin{aligned} \hat{k}(r_1, l_0) &= k(r_1, a_0(l_0), z_0) \\ &= \int_0^{z_1^*(a_0(l_0), z_0)} \left(\frac{\alpha z_1}{r_1 + \delta}\right)^{\frac{1}{1-\alpha}} dG_1(z) \\ &\quad + (1 - G_1(\hat{z}_1)) \lambda a_1(a_0(l_0), z_0), \end{aligned}$$

where the relationship between the initial net worth and leverage

$$a_0(l_0) = (1 - l_0)\left(\frac{\alpha z_0}{r_0 + \delta}\right)^{\frac{1}{1-\alpha}} \tag{20}$$

is obtained by rearranging (17) and the marginal unconstrained entrepreneur in the interim period

$$z_1^*(a_0(l_0), z_0) = \frac{r_1 + \delta}{r_0 + \delta} z_0 \lambda^{1-\alpha} \left[\frac{1-\alpha}{\alpha}(r_0 + \delta) + (1 + r_0)(1 - l_0)\right]^{1-\alpha}.$$

The last equation follows from (16), (19), and (20).

In this case, individuals with higher initial leverage are those with lower initial net worth. Therefore, highly leveraged individuals are those who are more likely to be constrained in the interim period. In particular, the fraction of unconstrained individuals in period $t = 1$ equals $G(z_1^*(a_0(l_0), z_0))$ and is a decreasing function of leverage l_0 as

$$\begin{aligned} \frac{\partial z_1^*}{\partial l_0} &= -(1-\alpha)(1 + r_0)\frac{r_1 + \delta}{r_0 + \delta} z_0 \lambda^{1-\alpha} \left[\frac{1-\alpha}{\alpha}(r_0 + \delta) + (1 + r_0)(1 - l_0)\right]^{-\alpha} \\ &< 0. \end{aligned}$$

The reduced form impact of initial leverage on the average sensitivity of period

$t = 1$'s capital input choice to a change in the interest rate r_1 is

$$\begin{aligned} \frac{\partial^2 \hat{k}_1 (r_1, l_0)}{\partial r_1 \partial l_0} &= \frac{\partial^2 k}{\partial r_1 \partial a_0} \frac{\partial a_0}{\partial l_0} \\ &= -\frac{1}{1-\alpha} (r_1 + \delta)^{-\frac{1}{1-\alpha}-1} (\alpha \hat{z}_1)^{\frac{1}{1-\alpha}} g(\hat{z}_1) \frac{\partial z_1^*}{\partial l_0} > 0. \end{aligned}$$

A change in the interest rate affects only the entrepreneurs whose investment is unconstrained, that is, entrepreneurs with relatively low productivity at time $t = 1$, i.e., $z_1 \leq z_1^*$. In the case in which leverage is driven by differences in the initial net worth, entrepreneurs who initially have higher leverage are more likely to be constrained and, therefore, they are less responsive to a change in the interest rate.

Heterogeneous z_0, Common a_0

We now consider the other extreme case, in which entrepreneurs have a common initial net worth and, therefore, the heterogeneity is only in terms of the initial productivity z_0. The reduced form relationship between investment, the interest rate shock, and initial leverage is

$$\begin{aligned} \hat{k}(r_1, l_0) &= k(r_1, a_0, z_0(l_0)) \\ &= \int_0^{z_1^*(a_0, z_0(l_0))} \left(\frac{\alpha z_1}{r_1 + \delta} \right)^{\frac{1}{1-\alpha}} dG_1(z) \\ &\quad + (1 - G_1(\hat{z}_1)) \lambda a_1 (a_0, z_0(l_0)), \end{aligned}$$

where the relationship between the initial productivity and leverage

$$z_0 = \frac{r_0 + \delta}{\alpha} \left(\frac{a_0}{1 - l_0} \right)^{1-\alpha} \tag{21}$$

is obtained by rearranging (17) and the marginal unconstrained entrepreneur in the interim period

$$z_1^*(a_0, z_0(l_0)) = \frac{r_1 + \delta}{\alpha} (\lambda a_0)^{1-\alpha} \left[\frac{1-\alpha}{\alpha} \frac{r_0 + \delta}{1 - l_0} + 1 + r_0 \right]^{1-\alpha}.$$

The last equation follows from (16), (19), and (21).

In this case, individuals with higher initial leverage are those with a higher initial productivity and, therefore, higher net worth at the beginning of the interim period.

Thus, highly leveraged individuals are those who are less likely to be constrained in the interim period. The fraction of unconstrained individuals in period $t = 1$ equals $G\left(z_1^*\left(a_0\left(l_0\right), z_0\right)\right)$ and is an increasing function of leverage l_0 as

$$\frac{\partial z_1^*}{\partial l_0} = \left(\frac{1-\alpha}{\alpha}\right)^2 \frac{\left(r_0+\delta\right)\left(r_1+\delta\right)}{\left(1-l_0\right)^2} \left(\lambda a_0\right)^{1-\alpha} \left[\frac{1-\alpha}{\alpha}\frac{r_0+\delta}{1-l_0} + 1 + r_0\right]^{-\alpha} > 0.$$

The reduced form impact of initial leverage on the average sensitivity of period $t = 1$'s capital input choice to a change in the interest rate r_1 is

$$\frac{\partial^2 \hat{k}_1\left(r_1, l_0\right)}{\partial r_1 \partial l_0} = -\frac{1}{1-\alpha}\left(\frac{1}{r_1+\delta}\right)^{\frac{1}{1-\alpha}+1} \left(\alpha \hat{z}_1\right)^{\frac{1}{1-\alpha}} g\left(\hat{z}_1\right) \frac{\partial z_1^*}{\partial l_0} < 0.$$

As before, a change in the interest rate affects only the entrepreneurs whose investment is unconstrained, that is, entrepreneurs with relatively low productivity at time $t = 1$, i.e., $z_1 \leq z_1^*$. In the case in which leverage is driven by differences in the initial productivity, entrepreneurs who are initially more leveraged are less likely to be constrained in the interim period and, therefore, they are more responsive to a change in the interest rate.

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